PROBLEMS IN LIST COLORING, TRIANGLE COVERING, AND PURSUIT-EVASION GAMES

BY

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DISSERTATION

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Abstract

We study several problems in extremal graph theory. Chapter 2 studies Tuza's Conjecture, which states that if a graph G does not contain more than k edge-disjoint triangles, then G can be made triangle-free by deleting at most $2k$ edges. Our results in Chapter 2 strengthen previous results on the conjecture, proving that the conjecture holds whenever G has no subgraph of average degree at least 7. Chapter 3 also deals with the problem of making a graph triangle-free, but from a different perspective: we consider a conjecture of Erdős, Gallai, and Tuza regarding "triangle independent" sets of edges. Writing $\tau_1(G)$ to denote the size of a smallest edge set X such that $G - X$ is triangle free and writing $\alpha_1(G)$ to denote the size of a largest edge set A that contains at most one edge from each triangle of G, the Erdős–Gallai–Tuza Conjecture states that $\alpha_1(G) + \tau_1(G) \leq |V(G)|^2/4$ for each graph G. We show that $\alpha_1(G) + \tau_1(G) \leq 5|V(G)|^2/16$; this improves on the trivial upper bound of $\binom{|V(G)|}{2}$. We also prove a general upper bound on $\alpha_1(G)$. In Chapter 4 we study multiple list coloring, which extends classical list coloring by requiring us to assign multiple colors to each vertex from its list. When L is a list assignment on a graph G, a *b-tuple coloring* of G assigns to each vertex v a set of b colors from $L(v)$ so that adjacent vertices receive disjoint sets of colors. Voigt conjectured that every bipartite minimal non-2-choosable graph is $(4:2)$-choosable. We disprove Voigt's conjecture, characterize which minimial non-2-choosable graphs are $(4:2)$-choosable, and conjecture an extension of Rubin's characterization of the 2-choosable graphs. Finally, in Chapter 5 we study the game of Revolutionaries and Spies, a pursuit-evasion game similar to Cops and Robbers. We use a probabilistic argument to analyze this game on hypercubes by relating it to a problem in extremal set theory, and we present winning strategies for the the spies on several classes of graphs.

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Table of Contents

Chapter 1

Introduction

In this thesis, we explore several problems in extremal graph theory. Chapter 2 studies a conjecture of Tuza concerning packing and covering of triangles, while Chapter 3 concerns a similar problem of Erdős, Gallai, and Tuza regarding "triangle independent" edge sets. Chapter 4 studies multiple list coloring. Finally, Chapter 5 presents several results on the the game of Revolutionaries and Spies.

In this chapter, we give an overview of our results. The last section, Section 1.5, gives formal definitions of many of the concepts used in the thesis, as well as presenting some tools used across different chapters, such as Hall's Theorem.

1.1 Tuza's Conjecture

Suppose that we wish to make a graph G triangle-free by deleting a small number of edges. An obvious obstruction is the presence of a large family of edge-disjoint triangles: we must delete one edge from each such triangle. On the other hand, deleting all edges from a maximal family of edge-disjoint triangles clearly destroys all triangles in G. Let $\nu(G)$ denote the maximum size of a set of edge-disjoint triangles in G, and let $\tau(G)$ denote the minimum size of an edge set Y such that $G - Y$ is triangle-free. We have just argued that $\nu(G) \le \tau(G) \le 3\nu(G)$. Clearly the lower bound is sharp, with equality in many instances, such as when all blocks are triangles. The desire to obtain also a sharp upper bound motivates the following conjecture:

Conjecture 1.1.1 (Tuza's Conjecture [33, 34])**.** *$\tau(G) \le 2\nu(G)$ for all graphs G.*

As Tuza noted, equality holds when all blocks are copies of K_4. Tuza showed that $\tau(G) \le 2\nu(G)$ when G is planar, and Krivelevich generalized this result by showing that $\tau(G) \le 2\nu(G)$ whenever G has no $K_{3,3}$-minor. Such graphs are "globally sparse", in the sense of the following definition:

Definition 1.1.2. The *maximum average degree* of a graph G, written $\text{Mad}(G)$, is defined by

$$\text{Mad}(G) = \max\left\{\frac{2\,|E(H)|}{|V(H)|}:\ H \subseteq G\right\}.$$

We prove the following stronger result:

Theorem 1.1.3. *If* $\mathrm{Mad}(G) < 7$*, then* $\tau(G) \leq 2\nu(G)$*.*

Several corollaries rely on the fact that their hypothesis guarantees $\mathrm{Mad}(G) < 7$:

Corollary 1.1.4 (Krivelevich)**.** *If G has no $K_{3,3}$-minor, then* $\tau(G) \leq 2\nu(G)$*.*

Corollary 1.1.5. *If G has no K_5-subdivision, then* $\tau(G) \leq 2\nu(G)$*.*

Corollary 1.1.6. *If G embeds in a torus, then* $\tau(G) \leq 2\nu(G)$*.*

In some sense, Theorem 1.1.3 can be interpreted as saying that Tuza and Krivelevich's results follow just from sparseness, rather than requiring topological properties.

1.2 The Erdős–Gallai–Tuza Conjecture

We again consider the problem of making a graph triangle-free, but from a different perspective. Changing notation slightly to maintain consistency with Erdős, Gallai, and Tuza [12], we let $\tau_1(G)$ denote the smallest number of edges that must be deleted to make G triangle-free, so that $\tau_1(G)$ is the same parameter as $\tau(G)$ in the previous section. Say that a set X of edges is a *hitting set* if $G - X$ is triangle-free.

A triangle edge cover contains *at least* one edge from each triangle in G. This suggests that we consider a dual notion: we say that a set of edges is *triangle-independent* if it contains *at most* one edge from each triangle in G, and write $\alpha_1(G)$ to denote the largest size of a triangle-independent set of edges.

Erdős [10] showed that every n-vertex graph G has a bipartite subgraph with at least $|E(G)|/2$ edges, which yields $\tau_1(G) \leq |E(G)|/2 < n^2/4$. Similarly, if A is triangle-independent, then the subgraph of G with edge set A is clearly triangle-free; by Mantel's Theorem, this implies that $\alpha_1(G) \leq n^2/4$.

Intuitively, $\alpha_1(G)$ and $\tau_1(G)$ cannot both be large: if $\tau_1(G)$ is close to $n^2/4$, then $|E(G)|$ is close to $n^2/2$, which makes it difficult to find a large triangle-independent set of edges. Erdős, Gallai, and Tuza formalized this intuition with the following conjecture.

Conjecture 1.2.1 (Erdős–Gallai–Tuza [12])**.** *For every n-vertex graph G,* $\alpha_1(G) + \tau_1(G) \leq n^2/4$*.*

The conjecture is sharp, if true: consider the graphs K_n and $K_{n/2,n/2}$, where n is even. We have $\alpha_1(K_n) = n/2$ and $\tau_1(K_n) = \binom{n}{2} - n^2/4$, while $\alpha_1(K_{n/2,n/2}) = n^2/4$ and $\tau_1(K_{n/2,n/2}) = 0$. In both cases, $\alpha_1(G) + \tau_1(G) = n^2/4$, but a different term dominates in each case. As observed by Erdős, Gallai, and Tuza, the difficulty of the conjecture lies in the variety of graphs for which the conjecture is sharp: any proof of the conjecture would need to account for both K_n and $K_{n/2,n/2}$ without any waste. In fact, we show that any graph G of the form $K_{r_1,r_1} \vee \cdots \vee K_{r_t,r_t}$ satisfies $\alpha_1(G) + \tau_1(G) = |V(G)|^2/4$; the graphs K_n and $K_{n/2,n/2}$ are the endpoints of this spectrum.

We obtain two partial results towards this conjecture. The first result is an upper bound on $\alpha_1(G)$ that is sharp for both K_n and $K_{n/2,n/2}$.

Theorem 1.2.2. *For an n-vertex graph G with m edges,*

$$\alpha_1(G) \leq \frac{n^2}{2} - m.$$

Equality holds if and only if there exist $r_1, \ldots, r_t \geq 1$ such that $G \cong K_{r_1,r_1,\ldots,r_t,r_t}$, that is, if G is a join of balanced complete bipartite graphs.

The second result is a general upper bound on $\alpha_1(G) + \tau_1(G)$. While it does not achieve the desired upper bound of $n^2/4$, it is a considerable improvement over the trivial upper bound of $n^2/2$.

Theorem 1.2.3. *For any graph G,*

$$\alpha_1(G) + \tau_1(G) \leq \frac{5n^2}{16.}$$

1.3 Multiple List Coloring

A *list assignment* on a graph G is a function L that gives every vertex v a set of colors $L(v)$. A graph G is *L-colorable* if it has a proper coloring f such that $f(v) \in L(v)$ for all v. If G is L-colorable whenever $|L(v)| \geq k$ for all v, then G is *k-choosable.* The *list chromatic number* of a graph G is the smallest k such that G is k-choosable. If G has list chromatic number k but all its proper subgraphs are $(k-1)$-choosable, we say that G is *k-choice-critical.*

Fractional choosability, also introduced by Erdős, Rubin and Taylor [15], generalizes ordinary choosability. When L is a list assignment on a graph G, a *b-tuple L-coloring* of G is a function f such that $f(v)$ is a b-element subset of $L(v)$ for all v. Such a coloring is *proper* if $f(v) \cap f(w) = \emptyset$ whenever v and w are adjacent. We say that G is *$(L:b)$-colorable* if G has a proper b-tuple L-coloring, and we say that G is *$(a:b)$-choosable* if G is *$(L:b)$-colorable* whenever $|L(v)| \geq a$ for all v.

Rubin (see [15]) characterized the 2-choosable graphs, and Tuza and Voigt [36] proved that all 2-choosable graphs are $(2m:m)$-choosable for all m. Voigt [39] proved that when m is odd, the 2-choosable graphs are the only $(2m:m)$-choosable graphs: that is, for any odd m, $(2m:m)$-choosability is equivalent to 2-choosability. On the other hand, when m is even, G can be $(2m:m)$-choosable and non-2-choosable: Tuza and Voigt [37] proved that $K_{2,4}$ is $(2m:m)$-choosable for all even m, while $K_{2,4}$ is not 2-choosable. Voigt [39] extended this result to show that many small 3-choice-critical graphs are $(2m : m)$-choosable when m is even, and conjectured that all bipartite 3-choice-critical graphs are $(2m : m)$-choosable when m is even.

A class of graphs called the *generalized theta graphs* play an important role in the characterization of the 2-choosable graphs. When $a_1, \ldots, a_t$ are nonnegative integers, the theta graph $\Theta_{a_1,\ldots,a_t}$ is obtained by starting with two vertices x and y and joining these vertices with t internally disjoint paths, the ith path having a_i edges.

In Chapter 4, we disprove Voigt's conjecture by determining which bipartite 3-choice-critical graphs are $(4 : 2)$-choosable. In particular, we prove that the 3-choice-critical graphs $\Theta_{2r,2s,2t}$ and $\Theta_{2r+1,2s+1,2t+1}$ fail to be $(4 : 2)$-choosable when $\min\{r, s, t\} \geq 2$. We also prove a weaker version of Voigt's Conjecture: there exists a (very large) m such that all bipartite 3-choice critical graphs are $(2m : m)$-choosable.

The results in Chapter 4 are joint work with Jixian Meng and Xuding Zhu.

1.4 Revolutionaries and Spies

The game of Revolutionaries and Spies is a pursuit-evasion game first defined by József Beck. The game is played on a graph G by a team of r revolutionaries and an opposing team of s spies; there is also a positive integer parameter m, called the *meeting size*. The rules of the game are as follows. At all times during the game, each revolutionary and each spy occupies a vertex of the graph. A vertex with m revolutionaries is a *meeting*; a meeting that also contains no spy is an *unguarded meeting*.

At the beginning of the game, each revolutionary chooses a vertex of the graph to occupy; multiple revolutionaries may occupy the same vertex. The spies then do the same. Play then proceeds as follows: if there is an unguarded meeting, then the revolutionaries win and the game is over. Otherwise, the game proceeds in *rounds*, a round consisting of the following steps:

- Each revolutionary has a chance to move, and may stay at its current vertex or move to a vertex adjacent to its current position.
- Each spy has a chance to move, in the same manner.
- At the end of the round, if there is still an unguarded meeting, then the game ends and the revolutionaries win. Otherwise, the game proceeds into a new round.

The revolutionaries win if there is ever an unguarded meeting at the end of a round, while the spies win by prolonging the game indefinitely.

We study this game by studying the function $\sigma(G, m, r)$, defined to be the smallest number of spies that win against r revolutionaries on the graph G with meeting size m. It is easy to show that

$$\min\{|V(G)|, \lfloor r/m \rfloor\} \le \sigma(G, m, r) \le r - m + 1$$

for any G and any r and m. We will therefore be particularly interested in when the actual value of $\sigma(G, m, r)$ is close to the lower bound – such circumstances can be considered "good" for the spies – and when $\sigma(G, m, r)$ is close to the upper bound, which can be considered "bad" for the spies.

A *dominating set* in a graph G is a set of vertices X such that every vertex in $V(G) - X$ has a neighbor in X. The *domination number* of a graph G is the size of a smallest dominating set in G. A result from [4] states that if G has domination number γ, then $\sigma(G, m, r) \le \gamma \lfloor r/m \rfloor$. In Section 5.3, we construct an infinite family of graphs for which this bound is sharp: that is, graphs for which $\gamma \lfloor r/m \rfloor - 1$ spies lose.

In Section 5.4, we study the behavior of the game on the d-dimensional hypercube graph. We show that when $d \ge r$, there is a constant c such that for all m, it takes $r - mc$ spies to defeat r revolutionaries. Thus, hypercubes are close to being as good as possible for the revolutionaries. The proof uses of a probabilistic argument applied to problem in extremal set theory that models the game.

In Section 5.5, we discuss some of the common ideas used to devise strategies for the spies. In Sections 5.6–5.8, we apply these ideas to several graph families. In Section 5.6, we show that if G has a spanning complete k-partite subgraph, then $\sigma(G, m, r) \le \left\lceil \frac{k}{k-1} \frac{r}{m} \right\rceil + k$. In Section 5.7, we consider the n-vertex random graph $G(n, p)$, where each possible edge is present with probability p.

In Section 5.8, we give a proof that was promised in the published paper [4]. A *split graph* is a graph whose vertices can be partitioned into an independent set and a clique. We show that if G is a split graph such that all the vertices in the independent set have degree at most d, then $\sigma(G, m, r) \le \lceil dr/m \rceil$: that is, the spies can use the structure of the split graph to their advantage, using what is in some sense a more sophisticated version of the dominating-set strategy.

The results in Chapter 5 are joint work with Jane V. Butterfield, Daniel W. Cranston, Douglas B. West, and Reza Zamani.

1.5 Definitions and Background

The *symmetric difference* of two sets X and Y, written $X \oplus Y$, is the set $(X \cup Y) - (X \cap Y)$. Symmetric differences have the property that $(X \oplus Y) \oplus Y = X$ for any sets X and Y. The size of the symmetric difference is a metric: for any sets X, Y, and Z, we have

1. $|X \oplus Y| \geq 0$, and $|X \oplus Y| = 0$ if and only if $X = Y$,

2. $|X \oplus Y| = |Y \oplus X|$, and

3. $|X \oplus Z| \leq |X \oplus Y| + |Y \oplus Z|$. (This is the *triangle inequality.*)

When X is a set and k is a nonnegative integer, we write $\binom{X}{k}$ for the family of all k-element subsets of X. A *graph* G is a pair $(V(G), E(G))$ where $V(G)$ is an arbitrary set and $E(G) \subseteq \binom{V}{2}$. The elements of $V(G)$ are called the *vertices* of G, and the elements of $E(G)$ are called the *edges* of G. When writing edges, we usually suppress set brackets and commas, writing uv instead of $\{u, v\}$.

Two vertices u and v are *adjacent* in a graph G if $uv \in E(G)$. We also write $u \leftrightarrow v$ to mean "u and v are adjacent". When v is a vertex of a graph G, the *neighborhood* of v in G is the set of all vertices adjacent to v. We write $N_G(v)$ for the neighborhood of v in G; when G is understood, we simply write $N(v)$, or speak of the "neighborhood of v" without qualification. The *degree* of v in G, written $d_G(v)$, is the size of $N_G(v)$; again, we suppress the subscript when G is understood. The *closed neighborhood* of v, written $N_G[v]$, is the set $N(v) \cup \{v\}$. The *Degree-Sum Formula* states that when G has finitely many edges, $\sum_{v \in V(G)} d(v) = 2\,|E(G)|$.

Formally, our definition of a graph allows the vertex set and edge set to be infinite. However, all graphs we consider in this thesis have a finite vertex set, and thus a finite edge set, unless otherwise specified.

When G is a graph, the *complement* of G, written $\overline{G}$ is the graph with vertex set $V(G)$ and edge set $\binom{V(G)}{2} - E(G)$. When G and H are graphs, the *union* of G and H, written $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. When $V(G)$ is disjoint from $V(H)$, their union is also written $G + H$ and called the *disjoint union* $G + H$. The notation $G + H$ is also used for graphs whose vertex sets are not disjoint; there, it represents the graph obtained by making $V(G)$ and $V(H)$ disjoint (say, by replacing each vertex $v \in V(G)$ with the tuple $(0, v)$ and replacing each $v \in V(H)$ with $(1, v)$) and then taking the union of the resulting graphs. The *join* of G and H, written $G \vee H$, is the graph obtained from $G + H$ by making every vertex from G adjacent to every vertex from H.

When G is a graph and $X \subseteq V(G)$, we sometimes write *X-vertex* to mean any vertex contained in X. Similarly, when $X \subseteq E(G)$, we write *X-edge* to refer to any edge contained in X.

An *isomorphism* from a graph G to a graph H is a bijection $f\colon\ G \to H$ such that, for any $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $f(u)f(v) \in E(G)$. Two graphs are *isomorphic* if there is an isomorphism from one to the other. When G and H are isomorphic, we write $G \cong H$. The class of graphs isomorphic to a given graph G is called the *isomorphism class* of G.

When k is a nonnegative integer, we write $[k]$ to denote the set consisting of the first k positive integers; note that $[0] = \emptyset$. For $n \geq 1$, the *path on n vertices* the graph with vertex set $[n]$ and edge set $\{\{i, i+1\} \colon i \in [n-1]\}$. The vertices of degree at most 1 are the *endpoints* of the path. For $n \geq 3$, a *cycle on n vertices* is the graph obtained from a path on n vertices by adding an edge joining its endpoints. An *even cycle* is a cycle with an even number of vertices, and an *odd cycle* is a cycle with an odd number of vertices. The *complete graph on n vertices* is the graph with vertex set $[n]$ where all pairs of vertices are adjacent. The isomorphism classes of n-vertex paths, cycles, and complete graphs are denoted by P_n, C_n, and K_n, respsectively.

When G and H are graphs, we say that H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. When G is a graph and $u, v \in V(G)$, a *u, v-path* is a subgraph of G that is a path with endpoints u and v. The *distance* between vertices u and v in a graph G, written $d_G(u, v)$, is the number of edges in a shortest u, v-path in G. When G is understood, we instead write $d(u, v)$. When $X \subseteq V(G)$, the *subgraph induced by X*, written $G[X]$, is the subgraph of G with vertex set X and edge set $E(G) \cap \binom{X}{2}$. When H is a graph, a graph G is *H-free* if it does not have an induced subgraph isomorphic to H. A *spanning subgraph* of a graph G is a subgraph H such that $V(G) = V(H)$.

A set of vertices X in a graph G is a *clique* if $G[X]$ is a complete graph; it is an *independent set* if $G[X]$ has no edges. The *independence number* of G, written $\alpha(G)$, is the largest size of an independent set in G.

When $X \subseteq V(G)$, the graph $G - X$ is the subgraph of G induced by $V(G) - V(X)$. When $X \subseteq E(G)$, the graph $G - X$ is the subgraph of G with vertex set $V(G)$ and edge set $E(G) - X$.

A graph G is *connected* if for any two vertices $u, v \in V(G)$, there is a u, v-path in G. When k is a positive integer, we say that G is *k-connected* if $|V(G)| > k$ and if $G - X$ is connected whenever X is a vertex set with $|X| \leq k - 1$. A *component* of a graph G is a maximal connected subgraph of G. A *block* of a graph G is a maximal 2-connected subgraph of G. A *cut vertex* of a graph G is a vertex v such that $G - v$ is not connected.

A graph G is *trivial* if it has exactly one vertex. A *forest* is a graph with that contains no cycle. A *tree* is a connected forest. A *leaf* in a forest is a vertex of degree 1. A nontrivial tree has at least two leaves. The *cycle rank* of a graph is the smallest size of an edge set X such that $G - X$ is a forest. It can be shown that when G has k components, the cycle rank of G is $|V(G)| - (n - k)$.

Given a graph G, one can define an auxiliary graph H as follows: the vertices of H are the blocks of G and the cut vertices of G, and a block b is adjacent to a cut vertex v when v belongs to the block b. (If G is 2-connected, then it has a single block and no cut vertices, so H is trivial.) It is not hard to show that H is always a forest, since a cycle in H yields a 2-connected subgraph in G that contains all the blocks in the cycle, contradicting the maximality of the blocks. Also, if G is connected, then H is a tree. A *leaf block* is a block in G that corresponds to a leaf in H, that is, a block in G that contains exactly one cut vertex. If G is connected but not 2-connected, then H is a nontrivial tree, and then G has a leaf block.

When k is a nonnegative integer, a *k-coloring* of a graph is any function $f : V(G) \to [k]$. In this context, we think of the elements of $[k]$ as *colors.* A *color class* for a given coloring is the set of all vertices receiving some fixed color. A k-coloring f is *proper* if $f(u) \neq f(v)$ whenever u and v are adjacent. The *chromatic number* of a graph G, written $\chi(G)$, is the least k such that there is a proper k-coloring of G.

A graph is *k-partite* if its chromatic number is at most k. Fixing some proper k-coloring of G, the color classes are called the *partite sets* of G. When $a_1, \ldots, a_k$ are positive integers, the *complete k-partite graph with part sizes* $a_1, \ldots, a_k$ is the graph $\overline{K_{a_1}} \vee \cdots \vee \overline{K_{a_k}}$. The isomorphism class of this graph is written $K_{a_1,\ldots,a_k}$. When the part sizes are not relevant, we may just say "complete k-partite graph".

A 2-partite graph is *bipartite*, and a bipartite graph with partite sets X and Y is an *X,Y-bigraph.* It is well known that a graph is bipartite if and only if it has no subgraph that is an odd cycle. A *complete bipartite graph* is a complete 2-partite graph. A *balanced complete bipartite graph* is a complete bipartite graph whose part sizes are equal.

A *matching* in a graph G is a set of pairwise disjoint edges of G. A matching M *covers* a vertex v if v lies in some edge of M; a matching covers a vertex set X if it covers every vertex in X. The *matching number* of a graph G, written $\alpha'(G)$, is the largest size of a matching in G.

A fundamental result in graph theory is Hall's Theorem, which gives necessary and sufficient conditions for the existence of a perfect matching in a bipartite graph:

Theorem 1.5.1 (Hall's Theorem [18])**.** *An X,Y-bigraph has a matching that covers X if and only if $|N(X_0)| \geq |X_0|$ for every $X_0 \subseteq X$.*

A function $f : X \to Y$ is *injective* if $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 are distinct elements of X. A function f is *surjective* if for every $y \in Y$ there is some $x \in X$ such that $f(x) = y$. A function f is *bijective* if it is both injective and surjective. A bijective function, or a *bijection*, is a function that is both injective and surjective. A bijection from a set to itself is a *permutation.*

When X is finite, a function $f\colon\ X \to X$ is injective if and only if it is surjective. Thus, to show that $f\colon\ X \to X$ is a permutation, it suffices to show that it is either injective or surjective.

A *fixed point* in a permutation f is an element a such that $f(a) = a$. A *k-cycle* in a permutation f is a list of elements $a_1, \ldots, a_k$ such that $f(a_i) = a_{i+1}$ for $i \in [k-1]$ and $f(a_k) = a_1$; note that a fixed point is a cycle containing one element. very permutation has a *disjoint cycle decomposition*, which partitions the domain of f into cycles; this decomposition is unique.

We now detail some probabilistic tools that we will need later. Although this tools are only used in Chapter 5, they require enough development that they would distract from the main line of the chapter there, so we develop them here instead. We assume that the basic terms of probability (expectation, independence, etc.) are known. Our treatment here follows the treatment given by Alon and Spencer [1].

A *Bernoulli random variable with success probability* p is a random variable that takes value 1 with probability p and takes value 0 with probability $1-p$. A *binomial random variable with* n *trials and success probability* p, written $\mathrm{Bin}(n,p)$, is a random variable that is the sum of n mutually independent Bernoulli random variables with success probability p.

Chernoff's Inequality roughly states that binomial random variables are tightly concentrated around their expectation. More broadly, it states that any variable which is the sum of independent Bernoulli variables is concentrated around its expectation:

Lemma 1.5.2 (Chernoff's Inequality [6]). *If X is a random variable that is the sum of n independent Bernoulli variables $X_1,\ldots,X_n$ with success probabilities $p_1,\ldots,p_n$, then for any positive real a,*

$$\begin{aligned}\mathbb{P}[X-\mathbb{E}[X]>a] &\le e^{-2a^2/n},\\ \mathbb{P}[X-\mathbb{E}[X]<-a] &\le e^{-a^2/(2pn)}, \qquad \textit{and}\\ \mathbb{P}[|X-\mathbb{E}[X]|>a] &\le 2e^{-2a^2/n}.\end{aligned}$$

where $p=(p_1+\cdots+p_n)/n$.

We will also need a result called the *FKG Inequality*, named after Fortuin, Kasteleyn, and Ginebre [16]. The actual statement of the FKG Inequality is quite technical, more than we need for this thesis; instead, we describe a corollary of the FKG Inequality that we will use later.

Suppose that we are generating a random subset X of $[n]$ by putting the element i into X with some fixed probability p_i, with all choices being made independently. A family of $\mathcal{A}$ of subsets of $[n]$ is a *downset* if $\mathcal{A}$ contains every subset of every set in $\mathcal{A}$. The following lemma, a corollary of the FKG Inequality, roughly says that if $\mathcal{A}$ and $\mathcal{B}$ are both downsets, then the events $X\in\mathcal{A}$ and $X\in\mathcal{B}$ are nonnegatively correlated:

Lemma 1.5.3. *If $\mathcal{A}$ and $\mathcal{B}$ are both downsets and X is generated as above, then*

$$\mathbb{P}[X\in\mathcal{A}\wedge X\in\mathcal{B}]\ge\mathbb{P}[X\in\mathcal{A}]\cdot\mathbb{P}[X\in\mathcal{B}].$$

Chapter 2

Tuza's Conjecture for Graphs with Maximum Average Degree less than 7

2.1 Introduction

Suppose that we wish to make a graph G triangle-free by deleting a small number of edges. An obvious obstruction is the presence of a large family of edge-disjoint triangles: we must delete one edge from each such triangle. On the other hand, deleting all edges from a maximal family of edge-disjoint triangles clearly destroys all triangles in G. Let $\nu(G)$ denote the maximum size of a set of edge-disjoint triangles in G, and let $\tau(G)$ denote the minimum size of an edge set Y such that $G - Y$ is triangle-free. We have just argued that $\nu(G) \leq \tau(G) \leq 3\nu(G)$. Clearly the lower bound is sharp, with equality in many instances, such as when all blocks are triangles. The desire to make the upper bound also sharp motivates the following conjecture:

Conjecture 2.1.1 (Tuza's Conjecture [33, 34])**.** *$\tau(G) \leq 2\nu(G)$ for all graphs G.*

Any graph whose blocks are all isomorphic to K_4 achieves equality in the upper bound, as observed by Tuza [34].

Tuza's Conjecture has been studied by many authors. The best general upper bound on $\tau(G)$ in terms of $\nu(G)$ is due to Haxell [21], who showed that $\tau(G) \leq 2.87\nu(G)$ for all graphs G.

Other authors have pursued the conjecture by showing that the desired bound $\tau(G) \leq 2\nu(G)$ holds for certain special classes of graphs. Tuza [34] showed that his conjecture holds for all planar graphs and for all K_6-free chordal graphs. Aparna Lakshmanan, Bujtás, and Tuza [3] generalized the result for planar graphs by showing that the conjecture holds for all "triangle-3-colorable" graphs, a class containing all 4-colorable graphs. Krivelevich [27] showed that Tuza's Conjecture holds for all graphs having no $K_{3,3}$-minor. The result on planar graphs was extended in a different direction by Haxell, Kostochka, and Thomassé [19], who proved that when G is a K_4-free planar graph, the stronger inequality $\tau(G) \leq \frac{3}{2}\nu(G)$ holds.

Krivelevich [27] also proved that a version of Tuza's Conjecture holds when τ or ν is replaced by its fractional relaxation τ^* or ν^*, where instead of asking for a set of edges Y or a set of edge-disjoint triangles $\mathcal{T}$, one instead asks for a *weight function* on the edges or the triangles of G, subject to constraints on the weight function which model the original constraints on Y and $\mathcal{T}$. Chapuy, DeVos, McDonald, Mohar, and Schiede [5] also studied a fractional version of Tuza's Conjecture, improving Krivelevich's bound of $\tau(G) \leq 2\nu^*(G)$ to $\tau(G) \leq 2\nu^*(G) - \frac{1}{\sqrt{6}}\sqrt{\nu^*(G)}$ and proving that this bound is tight. Chapuy, DeVos, McDonald, Mohar, and Schiede also extended Tuza's result on planar graphs, as well as Haxell's result, to the context of weighted graphs. Krivelevich's result was also extended by Haxell, Kostochka, and Thomassé [20], who proved a stability theorem: if $\tau^*(G) \geq 2\nu^*(G) - x$, then G contains a family of pairwise edge-disjoint subgraphs consisting of $\nu(G) - \lfloor 10x \rfloor$ copies of K_4 as well as $\lfloor 10x \rfloor$ triangles.

Haxell and Rödl [22] showed that if G is an n-vertex graph and $\nu^*(G)$ is the fractional relaxation of $\nu(G)$, then $\nu^*(G) - \nu(G) = o(n^2)$. As observed by Yuster [42], this result together with Krivelevich's result imply $\tau(G) \leq 2\nu(G) + o(n^2)$; thus, Tuza's Conjecture is asymptotically true for graphs containing a quadratic-sized family of edge-disjoint triangles. Such graphs are dense; instead, we study the conjecture on sparse graphs.

An important measure of sparseness is the *maximum average degree* of a graph, denoted $\mathrm{Mad}(G)$ and defined by

$$\mathrm{Mad}(G) = \max\left\{\frac{2\,|E(H)|}{|V(H)|} :\ H \subseteq G\right\}.$$

In this chapter, we apply the discharging method to prove the following theorem:

Theorem 2.1.2. *If* $\mathrm{Mad}(G) < 7$*, then* $\tau(G) \leq 2\nu(G)$*.*

To our knowledge, this is the first application of the discharging method to Tuza's Conjecture.

In Section 2.2 we introduce definitions and give the discharging argument used to prove Theorem 2.1.2, modulo two lemmas whose proof occupies most of the chapter. The key definition in Section 2.2 is that of a *reducible set*, a particular substructure that cannot occur in a smallest counterexample to Tuza's Conjecture. Essentially, a reducible set represents a "local solution" to the optimization problem posed by Tuza's Conjecture.

The definition of a reducible set for Tuza's Conjecture is perhaps the main new idea of the chapter. While we use discharging to prove the existence of reducible sets, we hope that later work will be able to use these reducible sets in extremal arguments which may not involve discharging at all.

In Section 2.3 we discuss some consequences of Theorem 2.1.2. In particular, we show that Theorem 2.1.2 implies that Tuza's Conjecture holds for toroidal graphs, for $K_{3,3}$-minor-free graphs, and for K_5-subdivision-free-graphs. We also discuss how to extend the result to graphs of genus at most 2.

In Sections 2.4–2.7 we prove the two lemmas stated in Section 2.2. In Section 2.4 we introduce *weak König–Egerváry graphs*, which we use heavily in our removability proofs. In Section 2.5 we discuss the behavior of low-degree vertices in graphs with no reducible set.

The results in Sections 2.2–2.5 are sufficient to prove a weaker result than Theorem 2.1.2. Using these results, we can show that Tuza's Conjecture holds for all graphs G with $\mathrm{Mad}(G) < 25/4$, a threshold which still suffices for many of the desired applications. In Section 2.6 we pause and sketch the proof of Tuza's Conjecture for graphs G with $\mathrm{Mad}(G) < 25/4$.

In Section 2.7 we explore the relation of *subsumption*, which plays a prominent role in the discharging rule of Section 2.2 and allows us to push the maximum average degree threshold up to 7. We again explore the behavior of this relation in graphs with no reducible set.

2.2 Definitions and Proof Summary

When G is a graph and $W \subseteq V(G)$, we write $G[W]$ for the subgraph of G induced by the vertices in W. When $V_0 \subseteq V(G)$, we write $N(V_0)$ for $\bigcup_{v \in V_0} N(v)$, and when $U \subseteq V(G)$, we write $N_U(V_0)$ for $N(V_0) \cap U$. Similarly, $d_U(V_0)$ denotes $|N_U(V_0)|$. We write K_n^- to denote the complete graph on n vertices with any edge removed. When the graph G is understood and k is a nonnegative integer, we say that a vertex of G is a *k-vertex* if its degree in G is exactly k, a *k^+-vertex* if its degree is at least k, or a *k^--vertex* if its degree is at most k.

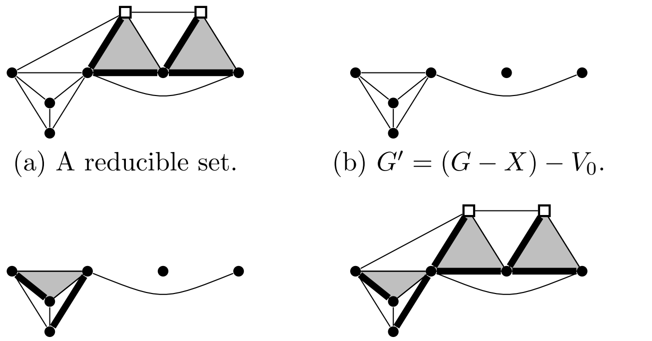
While Tuza's Conjecture involves two combinatorial optimization parameters, it can be also viewed as a single combinatorial optimization problem: in this problem, the goal is to simultaneously find a set $\mathcal{T}$ of edge-disjoint triangles and an edge set Y such that $G - Y$ is triangle-free and such that $|Y| \leq 2\,|\mathcal{T}|$.

The requirement that $G - Y$ be triangle-free is a global requirement. We replace this global problem with a local problem: fixing a vertex set V_0, we seek a set $\mathcal{S}$ of edge-disjoint triangles and an edge set X such that $G - X$ has no triangle containing a vertex of V_0 and such that $|X| \leq 2\,|\mathcal{S}|$. The rough idea is to remove the vertex set V_0, solve the "global" problem in the resulting subgraph, and then combine the subgraph solution with the "local solution" to solve the global problem in the original graph. The main difficulty in combining solutions this way is the requirement that the final set of triangles be edge-disjoint; carelessly combining sets of triangles will violate this requirement. The definition of a reducible set is tailored to overcome this difficulty:

Definition 2.2.1. When $\mathcal{S}$ is a set of triangles, an *$\mathcal{S}$-edge* is an edge of some triangle in $\mathcal{S}$. A nonempty set $V_0 \subseteq V(G)$ is *reducible* if there exist a set $\mathcal{S}$ of edge-disjoint triangles in G and set X of edges in G such that the following conditions hold:

(i) $|X| \leq 2\,|\mathcal{S}|$;

(ii) $G - X$ has no triangle containing a vertex of V_0; and



(a) A reducible set. (b) $G' = (G - X) - V_0$.

(c) $\mathcal{T}'$ and Y' in G'. (d) Combining $\mathcal{S}, X$ with $\mathcal{T}', Y'$.

Figure 2.1: Using a reducible set. Shaded triangles represent $\mathcal{S}$ and $\mathcal{T}'$; thick edges represent X and Y'; square white vertices represent V_0.

(iii) X contains every $\mathcal{S}$-edge whose endpoints are both outside V_0.

When V_0, $\mathcal{S}$, and X satisfy the definition above, we say that V_0 is *reducible using* $\mathcal{S}$ *and* X.

Note that Tuza's Conjecture holds for G if and only if the entire vertex set $V(G)$ is reducible. However, if G is a *minimal* counterexample to Tuza's Conjecture, then G has no reducible set of any size:

Lemma 2.2.2. *Let G be a graph, and let $V_0 \subseteq V(G)$ be reducible using $\mathcal{S}$ and X. Let $G' = (G - X) - V_0$. If $\tau(G') \leq 2\nu(G')$, then $\tau(G) \leq 2\nu(G)$.*

Proof. Let $\mathcal{T}'$ be a largest set of edge-disjoint triangles in G', and let Y' be a smallest set of edges such that $G' - Y'$ is triangle-free; by hypothesis, $|Y'| \leq 2\,|\mathcal{T}'|$. Let $\mathcal{T} = \mathcal{T}' \cup \mathcal{S}$ and $Y = Y' \cup X$. (The process is illustrated in Figure 2.1.) We show that $\mathcal{T}$ is a set of edge-disjoint triangles in G, that $G - Y$ is triangle-free, and that $|Y| \leq 2\,|\mathcal{T}|$, thus establishing the desired conclusion. The third condition is immediate from $|Y'| \leq 2\,|\mathcal{T}'|$ and $|X| \leq 2\,|\mathcal{S}|$.

To show that the triangles in $\mathcal{T}$ are pairwise edge-disjoint, it suffices to show that no $\mathcal{S}$-edge is a $\mathcal{T}'$-edge. This holds because every $\mathcal{T}'$-edge is contained in $(G - X) - V_0$, while every $\mathcal{S}$-edge is incident to V_0 or contained in X, by Condition (iii) of Definition 2.2.1.

Next we show that $G - Y$ is triangle-free. This holds because any triangle T in G satisfies one of the following three conditions:

(1) T is contained in $(G - X) - V_0$; or

(2) T contains a vertex of V_0; or

(3) T contains an edge of X.

Triangles of the first type meet Y', by hypothesis; triangles of the second type meet X, by Condition (ii) of Definition 2.2.1. $\square$

Our strategy for applying Lemma 2.2.2 is typical of discharging arguments: we show that various possible substructures of a graph G imply the existence of a reducible set, and we show that every graph with average degree less than 7 has one of these substructures. For more background on the discharging method, see [41].

To give the list of forbidden substructures, a few new definitions are needed:

Definition 2.2.3. A graph G is *robust* if for every $v \in V(G)$, every component of $G[N(v)]$ has order at least 5.

If G is robust, then $\delta(G) \geq 5$. Also, $G[N(v)]$ is connected whenever $d(v) < 10$.

Definition 2.2.4. A vertex u *subsumes* a vertex v if $N[u] \supseteq N[v]$.

Equivalently, u subsumes v if u is a dominating vertex in $G[N(v)]$.

Definition 2.2.5. A 6-vertex v is *thin* if $\overline{G[N(v)]}$ contains a matching of size 3.

The full list of forbidden substructures is given by the following two lemmas; the proof of the second lemma will occupy most of the chapter. For each part of the second lemma, we indicate which later results imply that part of the lemma.

Lemma 2.2.6. *If G is a minimal counterexample to Tuza's Conjecture, then G is robust.*

Lemma 2.2.7. *If G is robust and has no reducible set, then the following conditions hold.*

(a) Every 6^--vertex $v \in V(G)$ satisfies $\Delta(\overline{G[N(v)]}) \leq 1$ and $\left|E(\overline{G[N(v)]})\right| \neq 2$. (Proposition 2.5.1)

(b) The 6^--vertices of G form an independent set. (Proposition 2.5.2)

(c) No 7-vertex subsumes any 6-vertex. (Lemma 2.7.6)

(d) No 7-vertex is adjacent to any thin 6-vertex. (Lemma 2.7.6)

(e) No 8^--vertex subsumes any 5-vertex. (Lemma 2.7.6)

(f) Every 8-vertex subsuming a 6-vertex has at most three 6^--neighbors. (Lemma 2.7.5)

(g) Every 9-vertex subsumes at most three 6^--vertices, and a 9-vertex subsuming three 6^--vertices is adjacent to exactly three 6^--vertices. (Lemma 2.7.4)

(h) *Every 10^+-vertex v that subsumes some 6^--vertex has at most $d(v) - 6$ neighbors that are 6^--vertices. (Lemma 2.7.3)*

(i) *Every vertex v has at most $d(v) - 4$ neighbors that are 6^--vertices. (Corollary 2.5.3)*

Postponing the proof of Lemmas 2.2.6 and 2.2.7, we now give the proof of the main theorem.

Lemma 2.2.8. *Every robust graph with average degree less than 7 has a reducible set.*

Proof. Assuming that G has no reducible set, we use the method of discharging to show that G has average degree at least 7. Give every vertex v initial charge $d(v)$. We apply the following discharging rule:

- Every 5-vertex takes charge 2/3 from each vertex subsuming it;
- Every thin 6-vertex takes charge 1/6 from each neighbor;
- Every non-thin 6-vertex takes charge 1/4 from each vertex subsuming it.

We claim that every vertex has final charge at least 7, yielding average degree at least 7 in G.

First we consider the 6^--vertices. By part (b) of Lemma 2.2.7, no two such vertices are adjacent, so no 6^--vertex loses any charge when the discharging rule is applied. Thus we only need to check that each type of 6^--vertex gains enough charge to reach 7. There are no 4^--vertices, since G is robust. By part (a) of Lemma 2.2.7, every 5-vertex is subsumed by at least three vertices, and hence gains at least 2. Every thin 6-vertex gains exactly 1, for final charge 7. By part (a) of Lemma 2.2.7, the neighborhood of a non-thin 6-vertex v lacks at most one edge; hence v is subsumed by at least four vertices, and gains at least 1.

Now we consider the higher-degree vertices. Each 7-vertex starts with charge 7 and loses none, since it does not subsume any 5- or 6-vertices and is not adjacent to any thin 6-vertex, by parts (c)–(e) of Lemma 2.2.7.

Next, let v be an 8-vertex. By part (e) of Lemma 2.2.7, v does not subsume any 5-vertices. If v subsumes some 6-vertex w, then v subsumes at most three 6^--vertices, by part (f) of Lemma 2.2.7. Hence if v subsumes some 6-vertex, then v loses at most 3/4 charge, yielding final charge greater than 7. On the other hand, by part (i) of Lemma 2.2.7, v is adjacent to at most four 6^--vertices; hence, if v subsumes no 6-vertex, then v loses at most 4/6 charge, yielding final charge greater than 7.

Now, let v be a 9-vertex. By part (i) of Lemma 2.2.7, v has at most five 6^--neighbors in total. Hence, if v subsumes at most two 6^--vertices, then v loses at most $2(2/3) + 3(1/6)$ charge, yielding final charge greater than 7. On the other hand, if v subsumes three 6^--vertices, then by part (g) of Lemma 2.2.7 we see that v is adjacent to exactly those three 6^--vertices, so v loses exactly $3(2/3)$ charge, yielding final charge 7.

Finally, let v be a k-vertex with $k \geq 10$. If v subsumes no 6^--vertex, then v loses charge at most $k/6$, which yields final charge at least 7 since $k - k/6 \geq 7$. If v subsumes some 6^--vertex, then at most $k - 6$ neighbors of v are 6^--vertices, by part (h) of Lemma 2.2.7. Thus v loses at most $2(k-6)/3$, which yields final charge at least 7 since $k - 2(k-6)/3 \geq 7$. Hence all vertices have final charge at least 7, yielding average degree at least 7. ☐

Theorem 2.1.2. *If* $\mathrm{Mad}(G) < 7$*, then* $\tau(G) \leq 2\nu(G)$*.*

Proof. If the claim fails, let G be a minimal counterexample. Since $\mathrm{Mad}(G) < 7$, any proper subgraph G' also satisfies $\mathrm{Mad}(G') < 7$, so $\tau(G') \leq 2\nu(G')$ by the minimality of G. Thus, G is a minimal counterexample to Tuza's Conjecture among all graphs. By Lemma 2.2.6, G is robust, so by Lemma 2.2.8, G has a reducible set. Now Lemma 2.2.2 yields $\tau(G) \leq 2\nu(G)$, contradicting the choice of G as a counterexample. ☐

In the next section we explore some applications of Theorem 2.1.2 and its supporting lemmas. The remainder of the chapter will then be devoted to proving Lemma 2.2.7.

2.3 Consequences

Several earlier results on Tuza's Conjecture are natural consequences of Theorem 2.1.2. Tuza [34] proved that the conjecture holds for planar graphs. The following corollary of Euler's Formula extends the result to toroidal graphs, which are the graphs of genus at most 1:

Proposition 2.3.1. *If G is an n-vertex graph of genus γ with m edges, then $m \leq 3(n-2+2\gamma)$. In particular, G has average degree at most $6 + \frac{12(\gamma-1)}{n}$.*

Corollary 2.3.2. *If G is toroidal, then $\tau(G) \leq 2\nu(G)$.*

For higher genus, we obtain a finitization result.

Proposition 2.3.3. *For any fixed γ with $\gamma \geq 2$, if $\tau(G) \leq 2\nu(G)$ for all graphs G of genus at most γ with $|V(G)| \leq 12(\gamma-1)$, then $\tau(G) \leq 2\nu(G)$ for all graphs G of genus at most γ.*

Proof. Suppose not; let G be a minimal counterexample among the graphs of genus at most γ. All proper subgraphs G' also have genus at most γ, so they satisfy $\tau(G') \leq 2\nu(G')$, by the minimality of G. By hypothesis, $|V(G)| > 12(\gamma-1)$, so G has average degree less than 7. By Lemma 2.2.6, G is robust, so by Lemma 2.2.8, G has a reducible set. Thus $\tau(G) \leq 2\nu(G)$, by Lemma 2.2.2, contradicting the choice of G as a counterexample. ☐

For the case $\gamma = 2$, we performed an exhaustive computer search to verify the hypothesis of Proposition 2.3.3. By using Lemma 2.2.7, we avoid explicitly checking Tuza's Conjecture on graphs that can be shown to have reducible sets. Using the isomorph-free generation program `geng` [29] with a custom pruning function designed to recognize forbidden configurations (a) and (b) in Lemma 2.2.7, we identified a set of only 5299 graphs that contains any smallest counterexample of genus 2. (A database of these graphs, and tools for verifying the database, can be found at `http://www.math.uiuc.edu/~puleo/tuzaverify.tar.gz`) For higher γ, this computational approach quickly becomes intractible, even with such optimizations.

Krivelevich [27] proved Tuza's Conjecture for graphs with no $K_{3,3}$-minor. We obtain this from Theorem 2.1.2. Our proof relies on a theorem of Wagner ([40], described in [31]).

Definition 2.3.4. Let G_1 and G_2 be graphs. A *k-sum* of G_1 and G_2 is any graph obtained by identifying the vertices of a k-clique in G_1 with a k-clique in G_2 and then possibly deleting some edges of the merged k-clique. (In particular, a 0-sum is a disjoint union.)

Theorem 2.3.5 (Wagner [40]). *Any graph with no $K_{3,3}$-minor can be obtained by a sequence of* 0-,1-,*or* 2-*sums starting from planar graphs and K_5.*

Corollary 2.3.6. *If G is a graph with n vertices and no $K_{3,3}$-minor, then $|E(G)| \leq 3n - 5$.*

Proof. If G is planar or $G = K_5$, then the conclusion holds. It suffices to show that if G_1 and G_2 are graphs satisfying the bound, then any 0-, 1-, or 2-sum of G_1 and G_2 also satisfies the bound. This follows by straightforward algebra. In particular, for a j-sum of G_1 and G_2 with $j \in \{0, 1, 2\}$ and $n_i = |V(G_i)|$,

$$\begin{aligned}|E(G)| &\leq |E(G_1)| + |E(G_2)| - \binom{j}{2} \\ &\leq 3n_1 + 3n_2 - 10 - \binom{j}{2} \\ &= 3n - 5 + \left(3j - \binom{j}{2} - 5\right) \leq 3n - 5.\end{aligned}$$

□

Theorem 2.3.7 (Krivelevich). *If G is a graph with no $K_{3,3}$-minor, then $\tau(G) \leq 2\nu(G)$.*

Proof. Since all subgraphs of G also have no $K_{3,3}$-minor, Corollary 2.3.6 implies $\mathrm{Mad}(G) < 6$. □

Aparna Lakshmanan, Bujtás, and Tuza [3] proved that if G is 4-colorable, then $\tau(G) \leq 2\nu(G)$. This implies that Tuza's Conjecture holds for all graphs with no K_5-minor, since (as Wagner [40] showed) the Four-Color Theorem implies that all graphs with no K_5-minor are 4-colorable. Using a theorem of Mader [28] together with Theorem 2.1.2, we instead obtain the result for graphs with no K_5-subdivision:

Theorem 2.3.8. *If G is a graph with no K_5-subdivision, then $\tau(G) \le 2\nu(G)$.*

Proof. Since G has no K_5-subdivision, the number of edges in G is at most $3|V(G)| - 6$, as proved by Mader [28]. All subgraphs of G are K_5-subdivision-free, so $\mathrm{Mad}(G) < 6$. □

2.4 Weak König–Egerváry Graphs

A graph H is a *König–Egerváry graph* if $\alpha'(H) = \beta(H)$, where $\alpha'(H)$ is the matching number and $\beta(H)$ is the vertex cover number. The concept was introduced by Deming [8]; see also Kayll [25]. Let KE denote the class of König–Egerváry graphs. The König–Egerváry Theorem [9, 26] says that if H is bipartite, then $H \in \mathsf{KE}$. We weaken this definition, obtaining a larger class of graphs which will help streamline our removability proofs.

Definition 2.4.1. The graph H is a *weak König–Egerváry graph* if H has a matching M and a vertex set $Q \subseteq V(H)$ such that $|Q| \le |M|$ and Q is a vertex cover in $H - M$. Let WKE denote the class of weak König–Egerváry graphs. We say that a pair (M, Q) as above *witnesses* $H \in \mathsf{WKE}$.

Observe that $\mathsf{KE} \subseteq \mathsf{WKE}$: if $H \in \mathsf{KE}$, then (M, Q) witnesses $H \in \mathsf{WKE}$, where M is any maximum matching and Q is any minimum vertex cover in H.

To relate weak König–Egerváry graphs to reducible sets, we introduce an edge version of removability:

Definition 2.4.2. A nonempty edge set $E_0 \subseteq V(G)$ is *reducible* if there exist a set $\mathcal{S}$ of edge-disjoint triangles and a set X of edges of G such that the following conditions hold:

(i) $|X| \le 2|\mathcal{S}|$; and

(ii) $G - X$ has no triangle containing an edge of E_0; and

(iii) X contains every $\mathcal{S}$-edge that is not in E_0.

When E_0, $\mathcal{S}$, and X satisfy the definition above, we say that E_0 is *reducible using $\mathcal{S}$ and X*.

An analogue of Lemma 2.2.2 holds for reducible edge sets. The proof is essentially the same, so we do not repeat it here:

Lemma 2.4.3. *Let G be a graph, and let $E_0 \subseteq E(G)$ be reducible using $\mathcal{S}$ and X. Let $G' = (G - X) - E_0$. If $\tau(G') \le 2\nu(G')$, then $\tau(G) \le 2\nu(G)$.*

Lemma 2.4.4. *Let $v \in V(G)$, and let G_0 be a component of $G[N(v)]$. If $G_0 \in \mathsf{WKE}$, then G has a reducible set of edges. Also, if $G[N(v)] = G_0$ and $G_0 \in \mathsf{WKE}$, then $\{v\}$ is reducible in G.*

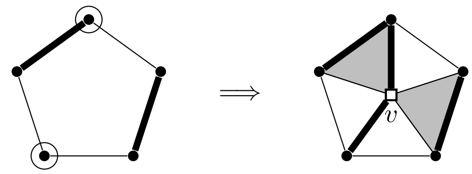


Figure 2.2: Transforming (M, Q) to $(\mathcal{S}, X)$.

Proof. Take any pair (M, Q) witnessing $G_0 \in \mathsf{WKE}$. Define E_0, $\mathcal{S}$, and X as follows:

$$E_0 = \{vw\colon\ w \in G_0\};$$

$$\mathcal{S} = \{vuw\colon\ uw \in M\};$$

$$X = M \cup \{vx\colon\ x \in Q\}.$$

Figure 2.2 illustrates the definition of $\mathcal{S}$ and X; in the figure, thick edges represent M and X, circled vertices represent Q, and shaded triangles represent $\mathcal{S}$.

Since M is a matching, the triangles in $\mathcal{S}$ are pairwise edge-disjoint. We claim that E_0 is reducible using $\mathcal{S}$ and X. Verifying each condition of Definition 2.4.2 in turn:

(i) Clearly $|X| \le 2\,|\mathcal{S}|$, since $|Q| \le |M|$.

(ii) Any triangle of G containing an edge of E_0 has the form vxy, where $xy \in E(H)$. Since Q is a vertex cover in $G_0 - M$, either $xy \in M$ or one of its endpoints lies in Q. Thus $G - X$ has no such triangle.

(iii) X contains every $\mathcal{S}$-edge that does not contain an edge of E_0, since $M \subseteq X$.

For the second claim, we observe that when E_0 is defined as above and $G_0 = G[N(v)]$, the condition $e \in E_0$ is equivalent to the condition $v \in e$. Comparing Definition 2.2.1 to Defintion 2.4.2, this shows that reducibility of E_0 is equivalent to reducibility of $\{v\}$. $\square$

Since all bipartite graphs are weak König–Egerváry graphs, Lemma 2.4.4 extends a theorem of Aparna Lakshmanan, Bujtás, and Tuza [3], who proved that if G is odd-wheel-free (i.e., locally bipartite) then Tuza's Conjecture holds for G. In the remainder of this section, we seek sufficient conditions for a graph to be a weak König–Egerváry graph. Due to Lemma 2.4.4, these conditions yield restrictions on the vertex neighborhoods in a minimum counterexample to Tuza's Conjecture.

The first such result is an analogue of the König–Egerváry Theorem: if H has no odd cycle of length greater than 3, then $H \in \mathsf{WKE}$. The proof relies on a characterization of such graphs due to Hsu, Ikura, and Nemhauser [24].

Theorem 2.4.5 (Hsu–Ikura–Nemhauser [24]). *If H is 2-connected and has no odd cycle of length greater than 3, then H is either bipartite, isomorphic to K_4, or isomorphic to $K_2 \vee \overline{K_r}$ for some $r \geq 1$.*

We start by proving a natural consequence of the König–Egerváry Theorem.

Lemma 2.4.6. *If v is a vertex in a bipartite graph H, then v lies in some minimum vertex cover of H if and only if v is covered by every maximum matching of H.*

Proof. If v lies in some minimum vertex cover Q, then $Q-v$ is a vertex cover of $H-v$, so $\beta(H-v) \leq \beta(H)-1$. By the König–Egerváry Theorem, $\alpha'(H-v) \leq \alpha'(H)-1$. This implies that v is covered by every maximum matching of H.

Now assume that v is covered in every maximum matching of H. This yields $\alpha'(H-v) = \alpha'(H)-1$. By the König–Egerváry Theorem, $\beta(H-v) = \alpha'(H)-1$. Adding v to a minimum vertex cover in $H-v$ yields the desired vertex cover. □

Next, we consider the nonbipartite graphs in Theorem 2.4.5, obtaining a stronger version of the weak König–Egerváry property:

Lemma 2.4.7. *If H is 2-connected and has no odd cycle of length greater than 3, then for every $v \in V(H)$, there is a vertex set Q and a matching M such that:*

1. *$|Q| \leq |M|$,*
2. *Q is a vertex cover in $H-M$, and*
3. *Either $v \in Q$ or v is in no edge of M.*

In particular, $H \in \mathsf{WKE}$.

Proof. If H is bipartite, then the claim follows from Lemma 2.4.6 together with the König–Egerváry Theorem: if v is covered by every maximum matching, then any minimum vertex cover has the desired properties. Thus, we may assume H is not bipartite. By Theorem 2.4.5, it suffices to consider three cases:

Case 1: $H \cong K_3$. Write $V(H) = \{v, w_1, w_2\}$. Let $Q = \{v\}$ and let $M = \{w_1w_2\}$; the only edge of H not covered by v is w_1w_2.

Case 2: $H \cong K_4$ *or* $H \cong K_2 \vee \overline{K_2}$. Either way, H has a matching M of size 2. Let Q be v together with its mate in M; the only edge of H not covered by Q is the other edge in M.

Case 3: $H \cong K_2 \vee \overline{K_m}$ *for* $m \geq 3$. Let Q consist of the two vertices of maximum degree. If $v \in Q$, then let M be any matching of size 2. Otherwise, $\alpha'(H-v) = 2$, so we can take M to be any maximum matching in $H-v$. □

Proposition 2.4.8. *If H has no odd cycle of length greater than 3, then $H \in \mathsf{WKE}$.*

Proof. We use induction on $|V(H)|$. If $|V(H)| = 1$ then clearly $H \in \mathsf{WKE}$. Now suppose that $|V(H)| > 1$ and the claim holds for all graphs with fewer vertices and no odd cycle of length greater than 3.

By the induction hypothesis, we may assume that H is connected. On the other hand, if H is 2-connected, then $H \in \mathsf{WKE}$, by Lemma 2.4.7. Thus we may assume that H is connected but not 2-connected, so H has a leaf block B. Note that $|V(B)| \geq 2$.

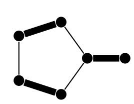
Let v be the cut vertex of H contained in B. By Lemma 2.4.7, B has a matching M_B and vertex cover Q_B such that $|Q_B| \leq |M_B|$, Q_B is a vertex cover in $B - M_B$, and either $v \in Q_B$ or v is in no edge of M_B. Note that $|V(B)| \geq 2$.

Now define a subgraph H' as follows: if $v \in Q_B$, let $H' = H - V(B)$; otherwise, let $H' = H - (V(B) - v)$. By the induction hypothesis, $H' \in \mathsf{WKE}$; let M' and Q' witness $H' \in \mathsf{WKE}$. Let $Q = Q' \cup Q_B$ and $M = M' \cup M_B$. Whether or not $v \in Q_B$, we see that M is a matching and that Q is a vertex cover in $H - M$. Clearly $|Q| \leq |M|$, so $H \in \mathsf{WKE}$. $\square$

Corollary 2.4.9. *If H is connected and $|V(H)| \leq 4$, then $H \in \mathsf{WKE}$.*

Corollary 2.4.10. *If H is connected and $\alpha'(H) \leq 1$, then $H \in \mathsf{WKE}$. Also, if H is connected, $|V(H)| > 5$, and $\alpha'(H) = 2$, then $H \in \mathsf{WKE}$.*

Proof. If $H \notin \mathsf{WKE}$, then H contains a cycle of length at least 5, which implies $\alpha'(H) \geq 2$. For the second claim, observe that any cycle of length at least 6 contains a matching of size 3, while if C is a 5-cycle in H, then there are adjacent vertices $v \in V(C)$ and $w \notin V(C)$, which yields the following matching of size 3:



$\square$

Recall that G is *robust* if for every $v \in V(G)$, every component of $G[N(v)]$ has order at least 5. In Section 2.2 we stated the following lemma, which now follows from the earlier results of this section:

Lemma 2.2.6. *If G is a minimal counterexample to Tuza's Conjecture, then G is robust.*

Proof. Follows immediately from Lemma 2.4.3, Lemma 2.4.4, and Corollary 2.4.9. $\square$

Corollary 2.4.11. *Let H be an n-vertex connected graph, where $n \geq 6$. If H has an independent set of size $n - 3$, then $H \in \mathsf{WKE}$.*

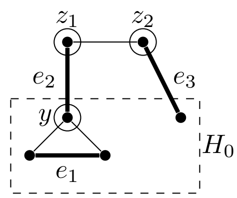


Figure 2.3: Finding a matching and vertex cover in H.

Proof. If $\alpha'(H) \leq 2$, then $H \in \mathsf{WKE}$, by Lemma 2.4.10. Otherwise, $\alpha'(H) \geq 3$, and the complement in $V(H)$ of a maximum independent set is a vertex cover of size at most 3. Thus $H \in \mathsf{KE}$. $\square$

Finally, we give a sufficient condition for small graphs to be weak König–Egerváry graphs. (In fact, the condition is also necessary, but we do not need the other direction, so we omit it.)

Proposition 2.4.12. *Let H be an n-vertex connected graph, where $n \in \{5, 6\}$. If $\Delta(\overline{H}) > 1$, then $H \in \mathsf{WKE}$.*

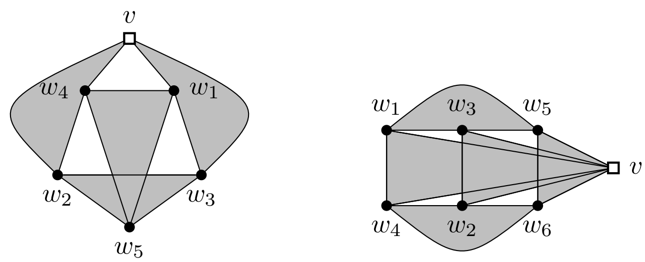
Proof. Since $\Delta(\overline{H}) > 1$, we may take $u, z_1, z_2 \in V(H)$ such that $uz_1, uz_2 \notin E(H)$. By Lemma 2.4.10, we may assume $\alpha'(H) = n - 3$, since $n \in \{5, 6\}$. If $z_1z_2 \notin E(H)$, then $V(H) - \{u, z_1, z_2\}$ is a vertex cover in H having size $n - 3$, which implies $H \in \mathsf{KE}$. Thus we may assume $z_1z_2 \in E(H)$. Also, if there is some maximum-size matching M containing the edge z_1z_2, then $V(H) - \{u, z_1, z_2\}$ is a vertex cover of size $n - 3$ in $H - M$, which implies $H \in \mathsf{WKE}$.

Case 1: *$n = 5$ and $\alpha'(H) = 2$.* Since no maximum-size matching contains z_1z_2, there are no edges in $H - \{z_1, z_2\}$, so $\{z_1, z_2\}$ is a vertex cover in H. Hence $H \in \mathsf{KE}$.

Case 2: *$n = 6$ and $\alpha'(H) = 3$.* Let $H_0 = H - \{z_1, z_2\}$; since no maximum-size matching contains z_1z_2, we have $\alpha'(H_0) \leq 1$. By Corollary 2.4.11, if H has an independent set of size 3, then $H \in \mathsf{WKE}$. Thus we may assume that $\alpha(H) < 3$, which implies that H_0 is a graph on 4 vertices such that $\alpha'(H_0) \leq 1$ and $\alpha(H_0) < 3$. This is only possible if $H_0 \cong K_3 + K_1$, as illustrated in Figure 2.3.

It follows that if M is any maximum-size matching in H, then one edge of M must lie inside the K_3 component of H_0, one edge must join the K_3 component to $\{z_1, z_2\}$, and one edge must join the K_1 component to $\{z_1, z_2\}$. Fix some maximum-size matching M and label its edges e_1, e_2, e_3 respectively. Let y be the vertex in the K_3 component not contained in e_1. The set $\{y, z_1, z_2\}$ is a vertex cover in $H - M$, so $H \in \mathsf{WKE}$. $\square$

Using weak König–Egerváry graphs, we have shown that any minimum counterexample to Tuza's Conjecture is robust (and therefore has minimum degree at least 5), and we have obtained strong restrictions on the possible neighborhoods of any 5- or 6- vertex.



(a) Case 1: $d(v) = 5$. (b) Case 2: $d(v) = 6$.

Figure 2.4: Triangles in Proposition 2.5.1.

2.5 Low-Degree Vertices

In this section, we will study the behavior of 6^--vertices in graphs with no reducible set. The main result of the section is Proposition 2.5.2, which states the 6^--vertices form an independent set; this result is used heavily in Section 2.7.

We first obtain a stronger version of Proposition 2.4.12, using the observation that it is possible for $\{v\}$ to be reducible even though $G[N(v)] \notin \mathsf{WKE}$.

Proposition 2.5.1. *Let G be a robust graph. If $v \in V(G)$ with $d(v) \leq 6$, then $\{v\}$ is reducible in G if $\Delta(\overline{G[N(v)]}) > 1$. Also, if $d(v) \leq 6$ and $\overline{G[N(v)]}$ has exactly 2 edges, then $\{v\}$ is reducible.*

Proof. The first statement follows immediately from Lemma 2.4.4 and Proposition 2.4.12. For the second statement, we again split into cases according to $d(v)$. Let $H = G[N(v)]$, and let $w_1, \ldots, w_{d(v)}$ be the vertices of H, indexed so that $E(\overline{H}) = \{w_1w_2, w_3w_4\}$.

Case 1: $d(v) = 5$. Define $\mathcal{S}$ and X by

$$\mathcal{S} = \{vw_2v_4, vw_1w_3, w_1w_4w_5, w_2w_3w_5\},$$
$$X = E(H).$$

The triangles in $\mathcal{S}$ are illustrated in Figure 2.4(a). We verify that $\{v\}$ is reducible using $\mathcal{S}$ and X, verifying each condition of Definition 2.2.1:

(i) $|X| \leq 2\,|\mathcal{S}|$, since $|E(H)| = 8$.

(ii) Every triangle containing v has its other two vertices in H, so $G - X$ has no triangle containing v.

(iii) X contains every $\mathcal{S}$-edge not incident to v, since all such edges lie in H.

Case 2: $d(v) = 6$. Define $\mathcal{S}$ and X by

$$\mathcal{S} = \{vw_1w_4, vw_2w_3, vw_5w_6, w_1w_3w_5, w_2w_4w_5\},$$
$$X = E(H - w_6) \cup \{w_5w_6, vw_6\}.$$

The triangles in $\mathcal{S}$ are illustrated in Figure 2.4(b). We verify that $\{v\}$ is reducible using $\mathcal{S}$ and X, verifying each condition of Definition 2.2.1:

(i) By construction, $|X| \leq 2\,|\mathcal{S}|$.

(ii) Since all edges of H not incident to w_6 lie in H, any triangle of $G - X$ containing v also contains w_6. Since $vw_6 \in X$, it follows that there is no such triangle.

(iii) By inspection, X contains every $\mathcal{S}$-edge that is not incident to v. □

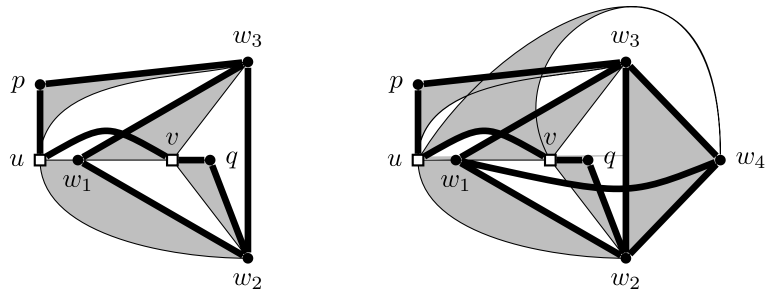
In the rest of the chapter, we will typically omit explicit verifications of Conditions (i) and (iii) of Definition 2.2.1, since they usually follow from a quick inspection of the definitions.

Next we show that a robust graph with no reducible set can have no edge joining "low-degree" vertices. The idea is simple: if u and v are adjacent low-degree vertices and neither $\{u\}$ nor $\{v\}$ is reducible, then we have a lot of information about the structure of $G[N(u)]$ and $G[N(v)]$, which will allow us to show that the set $\{u, v\}$ is reducible.

Proposition 2.5.2. *Let G be a robust graph. If $uv \in E(G)$ with $d(u) \leq 6$ and $d(v) \leq 6$, then one of $\{u\}$, $\{v\}$, or $\{u, v\}$ is reducible in G.*

Proof. Without loss of generality, we may assume $d(u) \leq d(v)$. Assuming that neither $\{u\}$ nor $\{v\}$ is reducible in G, we show that $\{u, v\}$ is reducible in G. Since neither $\{u\}$ nor $\{v\}$ is reducible, Proposition 2.5.1 says that $\Delta(\overline{G[N(u)]}) \leq 1$ and $\Delta(\overline{G[N(v)]}) \leq 1$. Let $H = G[N(u) \cap N(v)]$. Since $u, v \notin V(H)$ and u has at most one non-neighbor in $G[N(v)]$, we have $d(v) - 2 \leq |V(H)| \leq d(u) - 1$. Also, $\Delta(\overline{H}) \leq 1$, since $\Delta(\overline{G[N(u)]}) \leq 1$.

Case 1: $|V(H)| = 3$. In this case, $d(u) = d(v) = 5$ and v is not a dominating vertex in $G[N(u)]$. By Proposition 2.5.1, $G[N(u)]$ has precisely one non-edge, and likewise for $G[N(v)]$. Let p the unique vertex in $N(u) - N[v]$, and let q be the unique vertex in $N(v) - N[u]$; now pv is the unique non-edge in $G[N(u)]$ and qu is the unique non-edge in $G[N(v)]$. Write $V(H) = \{w_1, w_2, w_3\}$. Since $H \subseteq G[N(u)]$ and pv is the unique non-edge in $G[N(u)]$, we have $H \cong K_3$. Now $\{u, v\}$ is reducible using the following sets $\mathcal{S}$ and X,



(a) $\mathcal{S}, X$ in Case 1. (b) Largest possible $\mathcal{S}, X$ in Case 2.

Figure 2.5: Triangles and edges in Proposition 2.5.2.

illustrated in Figure 2.5(a):

$$\mathcal{S} = \{uw_1w_2, vw_1w_3, upw_3, vqw_2\}$$

$$X = \{uv, vq, up, pw_3, qw_2\} \cup E(H).$$

We quickly check Condition (ii) of Definition 2.2.1. Let T be a triangle in $G - X$ containing a vertex of V_0, say the vertex u. Since $E(H) \subseteq X$, at most one vertex of H lies in T, so T must contain a vertex in $\{v, p, q\}$. Since $uq \notin E(H)$ and $\{uv, up\} \subseteq X$, no such triangle exists. If T instead contains v, a similar argument holds.

Case 2: $|V(H)| = 4$. Since $\Delta(\overline{H}) \leq 1$, H contains incident edges w_1w_2 and w_1w_3; let w_4 be the remaining vertex of H. We build a set $\mathcal{S}$ of triangles and a set X of edges step-by-step as follows: initially, $\mathcal{S} = \{uw_1w_2, vw_1w_3, uvw_4\}$ and $X = E(H) \cup \{uv\}$. Note that initially, $2|\mathcal{S}| - |X| \geq -1$, with equality holding if and only if $H \cong K_4$. We augment $\mathcal{S}$ and X according to the following rules:

- If there exists $p \in N(u) - N[v]$, then add the triangle upw_3 to $\mathcal{S}$ and add the edges pu, pw_3 to X.
- If there exists $q \in N(v) - N[u]$, then add the triangle vqw_2 to $\mathcal{S}$ and add the edges qv, qw_2 to X.
- If $H \cong K_4$, add the triangle $w_2w_3w_4$ to $\mathcal{S}$.

Figure 2.5(b) shows the $\mathcal{S}$ and X obtained when p and q both exist and $H \cong K_4$. Note that if p exists, then p is the unique vertex of $N(u) - N[v]$, since v has at most one non-neighbor in $G[N(u)]$; likewise for q. In all cases, we end with $|X| \leq 2|\mathcal{S}|$. The verification of Condition (ii) is similar to Case 1.

Case 3: $|V(H)| = 5$. In this case, $d(u) = d(v) = 6$ and $N[u] = N[v]$. Since $\Delta(\overline{H}) \leq 1$, H contains a

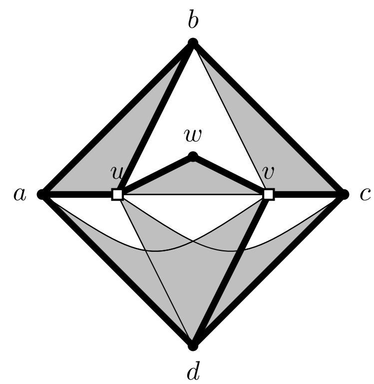


Figure 2.6: $\mathcal{S}, X$ in Case 3 of Proposition 2.5.2.

subgraph H' isomorphic to C_4, with vertices a, b, c, d in order. Let w be the remaining vertex of H. Now $\{u, v\}$ is reducible using the following sets $\mathcal{S}$ and X, illustrated in Figure 2.6:

$$\mathcal{S} = \{uab, ucd, vbc, vad, uvw\};$$

$$X = \{uw, vw, ua, ub, vc, vd\} \cup E(H').$$

We again check Condition (ii) of Definition 2.2.1. Any triangle containing the edge uv is of the form uvz, where $z \in V(H)$. For every such z, either $uz \in X$ or $vz \in X$; hence $G - X$ has no triangle containing uv. Now suppose T is a triangle containing u but not v. Clearly $a, b, w \notin T$; hence $T = ucd$, but $cd \in X$. Hence $G - X$ has no triangle containing u. A similar argument holds for v. □

Corollary 2.5.3. *Let G be a robust graph. If $v \in V(G)$ has more than $d(v)-4$ neighbors that are 6^--vertices, then $\{v\}$ is reducible.*

Proof. By Lemma 2.5.2, we may assume that $d(v) > 6$. Also by Lemma 2.5.2, the 6^--neighbors of v form an independent set in $G[N(v)]$. By Corollary 2.4.11, if $G[N(v)]$ has an independent set of size $d(v) - 3$, then $G[N(v)] \in \mathsf{WKE}$. Thus $\{v\}$ is reducible, by Lemma 2.4.4. □

2.6 A Weaker Result: $\mathrm{Mad}(G) < 25/4$

We now have sufficient tools to prove the following theorem, which is weaker than Theorem 2.1.2 but still strong enough for many of the applications in Section 2.3. In particular, this theorem is strong enough to imply Corollary 2.3.2 on toroidal graphs, Theorem 2.3.7 on $K_{3,3}$-minor-free graphs, and Theorem 2.3.8 on K_5-subdivision-free graphs.

Theorem 2.6.1. *If* $\mathrm{Mad}(G) < 25/4$*, then* $\tau(G) \le 2\nu(G)$*.*

Proof Sketch. Assuming that G has no reducible set, we use the method of discharging to show that G has average degree at least $25/4$. Give every vertex v initial charge $d(v)$. We apply the following discharging rule:

- Every 6^--vertex takes charge $1/4$ from every neighbor.

We claim that every vertex has final charge at least $25/4$, yielding average degree at least $25/4$ in G.

First we consider the 6^--vertices. By Lemma 2.2.6, G is robust, so $\delta(G) \ge 5$, and by Proposition 2.5.2, the 6^--vertices form an independent set. Hence all 5-vertices end with charge $25/4$, and all 6-vertices end with charge $30/4$.

Next we consider the 7^+-vertices. By Corollary 2.5.3, if v is a k-vertex where $k > 6$, then v has at most $k-4$ neighbors that are 6^--vertices. Hence v has final charge at least $3k/4 + 1$. Since $k \ge 7$, this implies that v has final charge at least $25/4$, as desired. $\square$

In the remaining section, we will improve the bound $\mathrm{Mad}(G) < 25/4$ to $\mathrm{Mad}(G) < 7$.

2.7 Subsumption and Related Bounds

Recall the following definitions from Section 2.2:

Definition 2.7.1. A vertex u *subsumes* a vertex v if $N[u] \supseteq N[v]$.

Definition 2.7.2. A 6-vertex v is *thin* if $\overline{G[N(v)]}$ contains a matching of size 3.

The motivation for these definitions is as follows: when u subsumes a 6^--vertex v, having $d(v)-1$ neighbors of v in $G[N(u)]$ leads to better bounds on the number of 6^--neighbors of u. Thus, in the discharging rule, such a vertex u can give away a lot of charge to the vertices it subsumes, since not many other 6^--neighbors will place demands on it. Conversely, if u subsumes no 6^--vertex, then the bounds on the number of 6^--neighbors are weaker, but since u does not subsume its neighbors, they need not demand much charge from u.

Lemma 2.7.3. *Let G be a robust graph with no reducible set. If a 10^+-vertex v subsumes a 6^--vertex w, then at most $d(v)-6$ neighbors of v are 6^--vertices.*

Proof. Assume to the contrary that $d(v)-5$ neighbors of v are 6^--vertices. We obtain a contradiction by proving $G[N(v)] \in \mathsf{WKE}$, implying that $\{v\}$ is reducible. Let A be the set of 6^--neighbors of v, and let

$B = N(v) - A$; note that $|A| \geq d(v) - 5 \geq 5 \geq |B|$. Since G has no reducible set, Proposition 2.5.2 implies that A is an independent set and that $N(w) - \{v\} \subseteq B$. Since A is independent, B is a vertex cover in $G[N(v)]$.

By Proposition 2.5.1, $\Delta(\overline{G[N(a)]}) \leq 1$ for all $a \in A$; in particular, since $v \in N(a)$, we have $d_{G[N(a)]}(v) \geq d(a) - 2$. Since $d_{G[N(a)]}(v) = |N(a) \cap N(v)| = d_{G[N(v)]}(a)$, we have $d_{G[N(v)]}(a) \geq d(a) - 2$. Since A is independent and each $d(a) \geq 5$, this implies $d_B(a) \geq 3$ for all $a \in A$. Similarly, $d_B(w) \geq 4$, since v is a dominating vertex in $G[N(w)]$ and $d(w) \geq 5$.

We first argue that $\alpha'(G[N(v)]) \geq 4$ by greedily constructing a matching of size 4. Let a_1, a_2, a_3 be distinct elements of $A - w$. Since each $d_B(a_i) \geq 3$, for each i we may choose $b_i \in N_B(a_i)$ distinct from all earlier b_i. Since $d_B(w) \geq 4$, we can take $b' \in B - \{b_1, b_2, b_3\}$. Now $\{a_1b_1, a_2b_2, a_3b_3, wb'\}$ is the desired matching of size 4. If $|B| = 4$, then this implies $G[N(v)] \in \mathsf{KE}$; thus we may assume $|B| = 5$. If $d_A(b) = 0$ for some $b \in B$, then $B - b$ is a vertex cover in $G[N(v)]$, which again implies $G[N(v)] \in \mathsf{KE}$; thus we may also assume $d_A(b) > 0$ for all $b \in B$.

Case 1: $\left|\bigcup_{z \in A-w} N_B(z)\right| = 3$. Let z_1, z_2, z_3 be distinct vertices in $A - w$, and let b_1, b_2, b_3 be distinct vertices in $\bigcup_{z \in A-w} N_B(z)$. Let $b' \in N_B(w) - \{b_1, b_2, b_3\}$. The set $B - b'$ is a vertex cover of size 4 in $G[N(v)] - \{wb', z_1b_1, z_2b_2, z_3b_3\}$, so $G[N(v)] \in \mathsf{WKE}$.

Case 2: $\left|\bigcup_{z \in A-w} N_B(z)\right| > 3$. We verify Hall's Condition for B. Take any $B_0 \subseteq B$. If $|B_0| > 2$, then $N_A(B_0) = A$, since each $a \in A$ has at most two non-neighbors in B. If $|B_0| = 1$, then $|N_A(B_0)| \geq 1$ since $d_A(b) > 0$ for all $b \in B$.

Now suppose $|B_0| = 2$. Since $d_B(w) \geq 4$, we have $w \in N_A(B_0)$. For $z \in A - w$, if $z \notin N_A(B_0)$, then $N_B(z) = B - B_0$, since $d_B(z) \geq 3$. Since $\left|\bigcup_{z \in A-w} N_B(z)\right| > 3$, the equality $N_B(z) = B - B_0$ cannot hold for all $z \in A - w$, so $|N_A(B_0)| \geq 2$. By Hall's Theorem, $\alpha'(G[N(v)]) \geq 5$, so $G \in \mathsf{KE}$. $\square$

When $d(v) = 9$ a similar statement holds, but more nuance is required, since we are no longer guaranteed that $|A| \geq |B|$.

Lemma 2.7.4. *Let G be a robust graph with no reducible set. Every 9-vertex subsumes at most three 6^--vertices; furthermore, if equality holds, then it is adjacent to no other 6^--vertex.*

Proof. Let v be a 9-vertex subsuming 6^--vertices w_1, w_2, w_3. Suppose to the contrary that v has another 6^--neighbor w' (possibly subsuming w', possibly not). Let $W = \{w_1, w_2, w_3, w'\}$, and let $V_0 = W \cup \{v\}$. We show that V_0 is reducible, contradicting the hypothesis. By Proposition 2.5.1, we have $\Delta(\overline{G[N(w')]}) \leq 1$, since G has no reducible set. Since $v \in N(w')$, this implies $|N(w') - N[v]| \leq 1$. By the definition of subsumption, $N(w_i) \subseteq N[v]$ for each i.

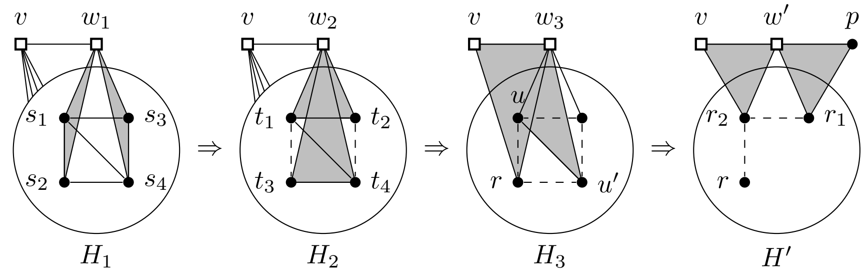


Figure 2.7: Constructing $\mathcal{S}$ in Lemma 2.7.4.

For convenience, let $H_i = G[N(w_i) \cap N(v)] = G[N(w_i) - \{v\}]$, and let $H' = G[N(w') \cap N(v)]$. By Proposition 2.5.2, the 6^--vertices of G form an independent set, so $V(H_i) \cap W = \emptyset$ for each i. We build a set $\mathcal{S}$ of edge-disjoint triangles in several steps; the observation that $V(H_i) \cap W = \emptyset$ helps guarantee that $\mathcal{S}$ is edge-disjoint. The algorithm begins with $S = \emptyset$.

(Figure 2.7 illustrates the construction in the "worst-case" scenario where $V(H_1) = V(H_2) = V(H_3)$, where each $H_i \cong K_4^-$, and where $V(H')$ is a proper subset of $V(H_1)$. In general, it is possible that the subgraphs H_i may have distinct vertex sets, but when they coincide we have less room to find edge-disjoint triangles. In the figure, dashed edges represent edges that are no longer available for use in $\mathcal{S}$, since they were used in earlier triangles.)

- Since $|V(H_1)| \geq 4$ and $\Delta(\overline{H_1}) \leq 1$, we can find two disjoint edges s_1s_2 and s_3s_4 in $E(H_1)$. Add the triangles $w_1s_1s_2$ and $w_1s_3s_4$ to $\mathcal{S}$.

- Since $E(\overline{H_2}) \cup \{s_1s_2, s_3s_4\}$ is the union of two matchings and $|V(H_2)| \geq 4$, we see that $H_2 - \{s_1s_2, s_3s_4\}$ is a graph on at least three edges that is neither a star nor a triangle. Hence there are two disjoint edges t_1t_2 and t_3t_4 in $E(H_2) - \{s_1s_2, s_3s_4\}$. Add the triangles $w_2t_1t_2$ and $w_2t_3t_4$ to $\mathcal{S}$.

- Since $\Delta(\overline{H_3}) \leq 1$ and $\left|E(\overline{H_3})\right| \neq 2$, and since $|V(H_3)| \geq 4$, we have $|E(H_3)| \geq 5$. Thus $H_3 - \{s_1s_2, s_3s_4, t_1t_2, t_3t_4\}$ still contains an edge uu' and a vertex $r \notin \{u, u'\}$. Add the triangles w_3uu' and vw_3r to $\mathcal{S}$.

- Since $\Delta(\overline{G[N(w')]}) \leq 1$, we have $|V(H')| \geq 3$. Fix any vertex $r_1 \in V(H') - \{r\}$ and add the triangle $vw'r_1$ to $\mathcal{S}$, reaching seven triangles. Also, if $N(w') - N[w] \neq \emptyset$, let p be the unique vertex in the difference. Note that $V(H') \subseteq N(p) \cap N(v)$, since otherwise p would have two non-neighbors in $G[N(w')]$, contradicting $\Delta(\overline{G[N(w')]}) \leq 1$. Choose $r_2 \in V(H') - \{r, r_1\}$ and add the triangle $w'r_2p$ to $\mathcal{S}$, reaching a total of eight triangles.

Figure 2.7 illustrates why the triangles in $\mathcal{S}$ are edge-disjoint. At each step, we add an edge-disjoint set of triangles, so it suffices to check that the triangles added in each step are disjoint from the earlier triangles. Since $V(H_i) \cap W = \emptyset$, edges incident to w_i are used only in the step corresponding to w_i; similarly, edges incident to w' are used only in the last step. By construction, we never use any edge in $E(H_i)$ that was previously used, so only the edges incident to v and incident to neither w_3 nor w' are liable to be reused. The only such edges are vr, vr_1, and possibly vr_2 and r_2p; since r, r_1, and r_2 were chosen to be distinct vertices, and since $p \notin N[v]$ while all other vertices used in $\mathcal{S}$ lie in $N[v]$, these edges are also distinct.

Let $Z = N(v) - V_0$, so that $|Z| = 5$. Define X by

$$X = \begin{cases} E(G[Z]) \cup \{vw_1, vw_2, vw_3, vw'\}, & \text{if } N(w') - N(v) = \emptyset; \\ E(G[Z]) \cup \{vw_1, vw_2, vw_3, vw', w'p, r_2p\}, & \text{if } N(w') - N(v) = \{p\}. \end{cases}$$

Since $|E(G[Z])| \leq 10$, we have $|X| \leq 2\,|\mathcal{S}|$ in either case. By construction, X contains every $\mathcal{S}$-edge that is not incident to V_0. We check that $G - X$ has no triangle containing a vertex of V_0. Since $E(G[Z]) \subseteq X$, any triangle in $G - X$ containing a vertex of V_0 must contain two vertices in $V_0 \cup (N(w') - N(v))$. Since W is an independent set, the only way for a triangle to contain two vertices in V_0 is to contain an edge of the form vw_i, vw', or $w'p$ if p exists. All such edges also lie in X; thus V_0 is reducible using $\mathcal{S}$ and X. $\square$

When $d(v) = 8$ and we are only concerned with 6-vertices, we obtain a similar result with a simple counting argument.

Lemma 2.7.5. *Let G be a robust graph with no reducible set. Every 8-vertex that subsumes a 6-vertex has most three 6^--neighbors.*

Proof. Let v be an 8-vertex that subsumes a 6-vertex w. By Proposition 2.5.2, w is not adjacent to any 6^--neighbor of v. Since $|N(v) \cap N(w)| = 5$, this implies that v has at most three 6^--neighbors. $\square$

Lemma 2.7.6. *Let G be a robust graph and let $uv \in E(G)$.*

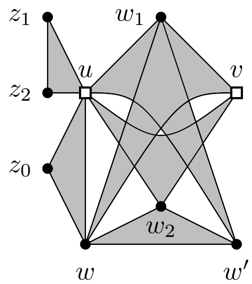
If $d(u) \in \{7, 8\}$ and $d(v) = 5$ and u subsumes v, then G has a reducible set;

If $d(u) = 7$ and $d(v) = 6$ and u subsumes v, then G has a reducible set;

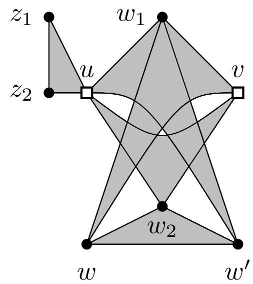
If $d(u) = 7$ and v is a thin 6-vertex, then G has a reducible set.

We prove each of these claims in its own proposition; the proofs are straightforward but require some case analysis.

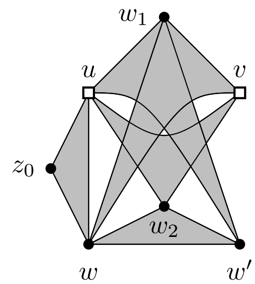
Proposition 2.7.7. *Let G be a robust graph. Let $uv \in E(G)$ with $d(u) \in \{7, 8\}$ and $d(v) = 5$. If neither $\{u\}$ nor $\{v\}$ is reducible in G and u subsumes v, then $\{u, v\}$ is reducible in G.*



(a) Case 1a.



(b) Case 1b.



(c) Case 2b(i).

Figure 2.8: Triangles in Proposition 2.7.7.

Proof. Let $W = N(u) \cap N(v)$ and $Z = N(u) - N[v]$. By Proposition 2.5.1, $G[N(v)] \in \{K_5, K_5^-\}$. Hence $G[W] \in \{K_4, K_4^-\}$. Also, $|Z| = d(u) - 5$, since u subsumes v. Hence $|Z| \in \{2, 3\}$.

If $G[W] \cong K_4^-$, then let w_1w_2 be the missing edge in W; otherwise, let w_1 and w_2 be distinct vertices of W.

Case 1: *$G[Z]$ contains an edge z_1z_2.* Let $Z^* = \{z \in Z\colon\ d_W(z) > 0\}$. Observe that $|Z^*| + |E(G[W])| \le 9$, with equality holding if and only if $|Z^*| = 3$ and $|E(G[W])| = 6$.

Case 1a: *$G[W] \cong K_4$ and $|Z^*| = 3$.* Let z_0 be the vertex of Z not in $\{z_1, z_2\}$. Choose $w \in N(z) \cap W$, relabeling if necessary so that w, w_1, w_2 are distinct, and let w' be the unique vertex in $W - \{w, w_1, w_2\}$. Now $\{u, v\}$ is reducible using the following sets $\mathcal{S}$ and X, with $\mathcal{S}$ illustrated in Figure 2.8(a):

$$\mathcal{S} = \{uw'w_1, vww_1, uvw_2, ww'w_2\} \cup \{uz_1z_2, uz_0w\},$$

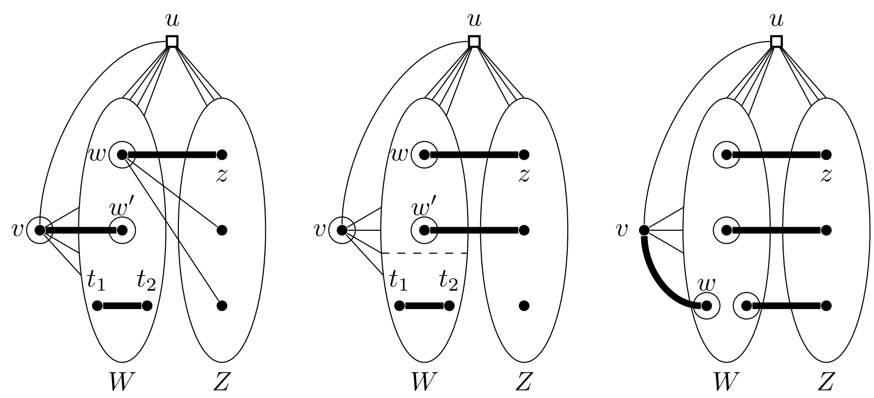
$$X = E(G[W]) \cup \{uz : z \in Z\} \cup \{uv, z_1z_2, z_0w\}.$$

We check Condition (ii) of Definition 2.2.1. Let T be a triangle in $G - X$ containing a vertex of V_0. Since $E(G[W]) \subseteq X$, we see that T contains at most one vertex from W; hence, two vertices of T must lie in $Z \cup \{u, v\}$. If $v \in T$, then T cannot contain any vertex of Z, so $\{u, v\} \subseteq T$, which is impossible since $uv \in X$. Therefore $v \notin T$, so T contains u and at least one vertex $z \in Z$. Since $uz \in X$, no such triangle exists.

Case 1b: $|Z^*| + |E(G[W])| \le 8$. Let w and w' be the vertices of $W - \{w_1, w_2\}$. If $|Z^*| < 2$, then enlarge Z^* to size 2 by adding arbitrary elements of Z. Now $\{u, v\}$ is reducible using the following sets $\mathcal{S}$ and X, with $\mathcal{S}$ illustrated in Figure 2.8(b):.

$$\mathcal{S} = \{uw'w_1, vww_1, uvw_2, ww'w_2\} \cup \{uz_1z_2\},$$

$$X = E(G[W]) \cup \{z_1z_2, uv\} \cup \{uz\colon\ z \in Z^*\}.$$



(a) Case 2a. (b) Case 2b(ii). (c) Case 2c.

Figure 2.9: Matchings and vertex covers in Case 2.

We again check Condition (ii) of Defintion 2.2.1. Let T be a triangle in $G-X$ containing a vertex of V_0. As before, T contains at most one vertex of W, and so $v \notin T$. Thus $u \in T$, so T either contains two vertices of Z or a vertex of Z and a vertex of W. Since $|Z^*| \geq 2$, if T contains two vertices of Z then T contains some vertex of Z^*, which is impossible since $uz \in X$ for all $z \in Z^*$. On the other hand, if T contains some $z \in Z$ and $w \in W$, then $d_W(z) > 0$, which implies $z \in Z^*$, again implying $uz \in X$.

Case 2: *Z is independent.* Since G is robust and $d(v) < 10$, $G[N(v)]$ is connected; thus no vertex of Z can be isolated in $G[N(v)]$, so each vertex of Z has a neighbor in W. Let J be the bipartite subgraph of G whose partite sets are W and Z. Since each vertex of Z has a neighbor in W, we have $\alpha'(J) > 0$.

Case 2a: $\alpha'(J) = 1$. Since every vertex of Z has a neighbor in W, $\alpha'(J) = 1$ implies that some vertex $w \in W$ covers every edge incident to Z. Let z be any vertex of Z, let t_1t_2 be an edge in $G[W]$ not containing w, and let w' be the remaining vertex of W. Let $M = \{wz, t_1t_2, uw'\}$ and let $Q = \{v, w, w'\}$, as illustrated in Figure 2.9(a). All edges incident to Z are covered by W, and the only edge of $G[W]$ not covered by $\{w, w'\}$ is t_1t_2, so Q is a vertex cover in $G[N(u)] - M$. Hence $G[N(u)] \in \mathsf{WKE}$, contradicting the hypothesis that $\{u\}$ is not reducible.

Case 2b: $\alpha'(J) = 2$. Since $\alpha'(J) = 2$, $|N_W(Z)| \geq 2$.

Case 2b(i): $|N_W(Z)| \geq 3$. Let $w \in N_W(Z) - \{w_1, w_2\}$, and pick $z_0 \in Z$ such that $wz \in E(G)$. Let w' be the unique vertex in $W - \{w, w_1, w_2\}$, and let $\{q_1, q_2\}$ be a vertex cover in J. Now $\{u, v\}$ is reducible using the following sets $\mathcal{S}$ and X, with $\mathcal{S}$ illustrated in Figure 2.8(c):

$$\mathcal{S} = \{uw'w_1, vww_1, uvw_2, ww'w_2\} \cup \{uz_0w\},$$

$$X = E(G[W]) \cup \{uv, z_0w\} \cup \{uq_1, uq_2\}.$$

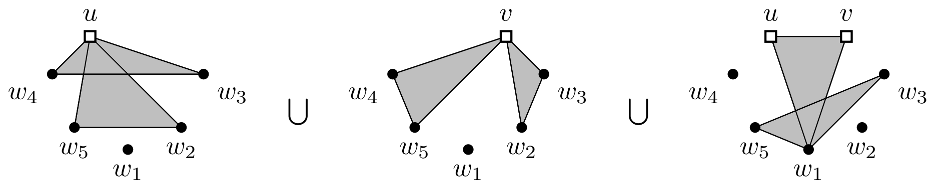


Figure 2.10: Triangles in Proposition 2.7.8.

The verification of Condition (ii) is similar to Case 1b, with the following modifications: any bad triangle T cannot contain two vertices of Z, since $G[Z]$ has no edges; and if T contains a vertex in Z and a vertex in W, then T contains u along with an edge in J, one endpoint of which lies in $\{q_1, q_2\}$. The other possibilities for T are identical to Case 1b.

Case 2b(ii): $|N_W(Z)| = 2$. Let t_1, t_2 be the two vertices of $W - N_W(Z)$, and let M be a maximum matching in J, as illustrated in Figure 2.9(b) in the case where $t_1t_2 \in E(G)$. Clearly, M does not cover t_1 or t_2. Observe that $N_W(Z) \cup \{v\}$ covers every edge in $G[N(u)]$, except possibly the edge t_1t_2 if it exists. If $t_1t_2 \in E(G)$, then let $M' = M \cup \{t_1t_2\}$; otherwise, let $M' = M \cup \{vt_1\}$. In either case, $N_W(Z) \cup \{v\}$ is a vertex cover of size 3 in $N(v) - M'$, so $G[N(u)] \in \mathsf{WKE}$, contradicting the hypothesis that $\{u\}$ is not reducible.

Case 2c: $\alpha'(J) = 3$. Let M be a maximum matching in J, and let w be the vertex of M not covered by M; then $M \cup \{vw\}$ is a matching of size 4 in $G[N(u)]$, as shown in Figure 2.9(c). Since W is a vertex cover of size 4 in $G[N(u)]$, this implies $G[N(u)] \in \mathsf{KE}$, again contradicting the hypothesis that $\{u\}$ is not reducible. $\square$

Proposition 2.7.8. *Let G be a robust graph. Let $uv \in E(G)$ with $d(u) = 7$ and $d(v) = 6$. If $\{u\}$ is not reducible in G and u subsumes v, then $\{u, v\}$ is reducible in G.*

Proof. By Proposition 2.5.1, $G[N(v)] \in \{K_6, K_6^-\}$ (since $G[N(v)]$ has u as a dominating vertex, $\overline{G[N(v)]}$ cannot be a perfect matching). Let $H = G[N(u) \cap N(v)]$ and write $V(H) = \{w_1, \ldots, w_5\}$, indexed so that w_1w_2 is the possible missing edge. Let p be the unique vertex in $N(u) - N[v]$. Now $\{u, v\}$ is reducible using the following sets $\mathcal{S}$ and X, with $\mathcal{S}$ illustrated in Figure 2.10:

$$\mathcal{S} = \{uw_2w_5, uw_3w_4\} \cup \{vw_2w_3, vw_4w_5\} \cup \{uvw_1, w_1w_3w_5\},$$
$$X = E(H) \cup \{uv, up\}.$$

We check Condition (ii) of Definition 2.2.1. Since $E(H) \subseteq X$, any triangle of $G - X$ containing a vertex of V_0 contains at most one vertex of H, and therefore contains two vertices from $\{u, v, p\}$. Since $uv, up \in X$

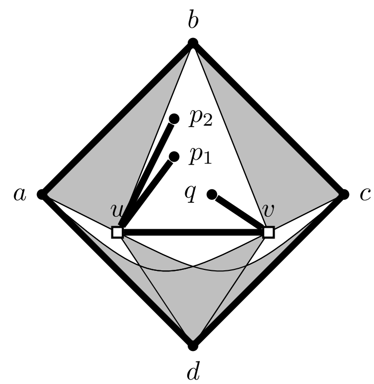


Figure 2.11: $\mathcal{S}, X$ in Proposition 2.7.9.

and $vp \notin E(G)$, no such triangle exists. $\square$

Proposition 2.7.9. *Let G be a robust graph. If G contains an edge uv such that $d(u) = 7$ and v is a thin 6-vertex, then $\{u, v\}$ is reducible in G.*

Proof. Since $\overline{G[N(v)]}$ is a matching of size 3, we know that $G[N(u) \cap N(v)] \cong C_4$; let a, b, c, d be the vertices of this cycle, listed in order. Let p_1, p_2 be the two vertices of $N(u) - N[v]$ and let q be the unique vertex in $N(v) - N[u]$. Now $\{u, v\}$ is reducible using the following sets $\mathcal{S}$ and X, illustrated in Figure 2.11:

$$\mathcal{S} = \{uab, ucd, vbc, vad\},$$

$$X = E(G[N(u) \cap N(v)]) \cup \{uv, up_1, up_2, vq\}.$$

We quickly check Condition (ii) of Definition 2.2.1. Since $E(G[N(u) \cap N(v)]) \subseteq X$, any triangle of $G - X$ containing a vertex of V_0 contains at most one vertex from $N(u) \cap N(v)$, and therefore contains two vertices from $\{u, v, p_1, p_2, q\}$. Let T be a triangle of $G - X$ and suppose $u \in T$. Since $uv, up_1, up_2 \in X$ and $uq \notin E(G)$, we see that T cannot contain two vertices of $\{u, v, p_1, p_2, q\}$, and so $G - X$ has no triangle containing u. Similar logic holds for v. $\square$

We have now completed the proof of Lemma 2.2.7, giving a list of configurations that cannot appear in a smallest counterexample to Tuza's Conjecture. By completing the proof of this lemma, we have completed the proof of the main theorem, Theorem 2.1.2.

Chapter 3

On a Conjecture of Erdős, Gallai, and Tuza

3.1 Introduction

Given an n-vertex graph G, say that a set $A \subseteq E(G)$ is *triangle-independent* if it contains at most one edge from each triangle of G, and say that $X \subseteq E(G)$ is a *hitting set* if $G - X$ is triangle-free. (The name "hitting set" comes from the fact that such an edge set "hits" every edge in the 3-uniform hypergraph with vertex set $E(G)$ whose edges are the edge sets of triangles in G.) Throughout this chapter, $\alpha_1(G)$ denotes the maximum size of a triangle-independent set of edges in G, while $\tau_1(G)$ denotes the minimum size of a hitting set in G.

Mantel's Theorem, proved in 1907, states that any n-vertex triangle-free graph has at most $n^2/4$ edges. Mantel's Theorem is a special case of Turán's Theorem, which states that for $r \geq 1$, any n-vertex K_r-free graph has at most $\frac{r-1}{r}\frac{n^2}{2}$ edges [32]. When $r-1$ divides n, this is the number of edges in a balanced complete $(r-1)$-partite graph on n vertices.

Erdős [10] showed that every n-vertex graph G has a bipartite subgraph with at least $|E(G)|/2$ edges, which implies that $\tau_1(G) \leq |E(G)|/2 \leq (n^2-n)/4$. Similarly, if A is triangle-independent, then the subgraph of G with edge set A is clearly triangle-free; by Mantel's Theorem, this implies that $\alpha_1(G) \leq n^2/4$.

Intuitively, $\alpha_1(G)$ and $\tau_1(G)$ cannot both be large: if $\tau_1(G)$ is close to $n^2/4$, then $|E(G)|$ is close to $n^2/2$, which makes it difficult to find a large triangle-independent set of edges. Erdős, Gallai, and Tuza formalized this intuition with the following conjecture.

Conjecture 3.1.1 (Erdős–Gallai–Tuza [12]). *For every n-vertex graph G, $\alpha_1(G) + \tau_1(G) \leq n^2/4$.*

The conjecture is sharp, if true: consider the graphs K_n and $K_{n/2,n/2}$, where n is even. We have $\alpha_1(K_n) = n/2$ and $\tau_1(K_n) = \binom{n}{2} - n^2/4$, while $\alpha_1(K_{n/2,n/2}) = n^2/4$ and $\tau_1(K_{n/2,n/2}) = 0$. In both cases, $\alpha_1(G) + \tau_1(G) = n^2/4$, but a different term dominates in each case. As observed by Erdős, Gallai, and Tuza, the difficulty of the conjecture lies in the variety of graphs for which the conjecture is sharp: any proof of the conjecture would need to account for both K_n and $K_{n/2,n/2}$ without any waste.

Erdős, Gallai, and Tuza [12] considered the conjecture on graphs for which every edge lies in a triangle, and proved that there is a positive constant c such that $\alpha_1(G)+\tau_1(G) \le |E(G)|-c\,|E(G)|^{1/3}$ and $\alpha_1(G)+\tau_1(G) \le |E(G)| - c\,|V(G)|^{1/2}$ for such graphs. Aside from the original paper of Erdős, Gallai, and Tuza, no other work appears to have been done on the conjecture.

In this chapter, we present two partial results towards Conjecture 3.1.1.

In Section 3.2, we extend some ideas of Erdős, Faudree, Pach and Spencer [11] in order to obtain the bound $\alpha_1(G) + \tau_1(G) \le 5n^2/16$. To our knowledge, this is the best general bound on $\alpha_1(G) + \tau_1(G)$.

In Section 3.3, we obtain the bound $\alpha_1(G) \le n^2/2 - m$, where $m = |E(G)|$, and characterize the graphs for which equality holds. When n is even, this bound is sharp for both K_n and $K_{n/2,n/2}$, which makes it an encouraging step towards the Erdős–Gallai–Tuza Conjecture.

3.2 Induced Bipartite Subgraphs

In this section, we will focus on the relationship between triangle-free subgraphs of G and bipartite subgraphs of G. The problem of finding a largest bipartite subgraph of a graph G is well-studied, and clearly any bipartite subgraph of G is triangle-free, so we can reasonably hope to apply some of the existing literature to our current problem.

Erdős, Faudree, Pach, and Spencer [11] studied the problem of finding a largest bipartite subgraph of a triangle-free graph, using the observation that if uv is an edge of a triangle-free graph, then $G[N(u) \cup N(v)]$ is an induced bipartite subgraph of G. Here, we use a similar observation: if A is a triangle-independent set of edges in G, then for any $uv \in E(G)$, the induced subgraph $G[N_A(u) \cup N_A(v)]$ is bipartite (even when the edges in $E(G) - A$ are considered). The point is that A is not only triangle-free itself, but imposes restrictions on the triangles in G.

We define some useful notation. For any graph G, let $\tau'(G)$ denote the smallest size of an edge set X such that $G - X$ is bipartite, and let $b(G)$ denote the maximum size of a vertex set B such that $G[B]$ is bipartite. Clearly, $\tau_1(G) \le \tau'(G)$, so we seek bounds on $\alpha_1(G) + \tau'(G)$. When $A \subseteq E(G)$, we will abuse notation by identifying A with the spanning subgraph of G having edge set A. This yields notation like $N_A(v)$, referring to the neighborhood of a vertex v in the spanning subgraph of G with edge set A.

Lemma 3.2.1. *For any graph G, any triangle-independent set $A \subseteq E(G)$, and any edge $uv \in E(G)$,*

$$d_A(u) + d_A(v) \le b(G).$$

Proof. Since A is triangle-independent, the sets $N_A(u)$ and $N_A(v)$ are independent and disjoint. Hence

$G[N_A(u) \cup N_A(v)]$ is bipartite with $d_A(u) + d_A(v)$ vertices. $\square$

Lemma 3.2.2. *For any graph G,*

$$\alpha_1(G) \leq \frac{nb(G)}{4.}$$

Proof. Let A be any triangle-independent subset of $E(G)$. Applying Lemma 3.2.1 to all edges in A and summing the resulting inequalities gives

$$\sum_{u \in V(G)} d_A(u)^2 = \sum_{uv \in A} (d_A(u) + d_A(v)) \leq |A|\, b(G).$$

By the Cauchy-Schwarz Inequality, we have

$$\sum_{u \in V(G)} d_A(u)^2 \geq \frac{4\,|A|^2}{n.}$$

The desired inequality follows. $\square$

Lemma 3.2.3. *For any graph G,*

$$\tau'(G) \leq \frac{n^2}{4} - \frac{b(G)^2}{4.}$$

Proof. This is essentially the case $\delta = 0$ of Proposition 2.5 of [11]. We sketch a probabilistic proof here. Let B be a largest vertex set such that $G[B]$ is bipartite. If we randomly add each vertex of $V(G) - B$ to one partite sets of B, the expected number of edges with both endpoints in the same partite set is $(E(G) - E(G[B]))/2$; hence G can be made bipartite by deleting at most this many edges. Hence

$$\tau'(G) \leq \frac{1}{2}\,|E(G) - E(G[B])| \leq \frac{1}{2}\left(\binom{n}{2} - \binom{b(G)}{2}\right) \leq \frac{n^2}{4} - \frac{b(G)^2}{4.} \qquad \square$$

Corollary 3.2.4. *For any graph G,*

$$\alpha_1(G) + \tau'(G) \leq \frac{5n^2}{16.}$$

Proof. From Lemma 3.2.2 and Corollary 3.2.3, we immediately have

$$\alpha_1(G) + \tau'(G) \leq \frac{n^2}{4} + \frac{nb(G)}{4} - \frac{b(G)^2}{4.}$$

Since the function $x(n - x)$ is maximized when $x = n/2$, this implies

$$\alpha_1(G) + \tau'(G) \leq \frac{5n^2}{16,}$$

as desired. $\square$

3.3 Bounding $\alpha_1(G)$

In this section, we will obtain the bound $\alpha_1(G) \leq n^2/2 - m$. We first need one quick lemma.

Lemma 3.3.1. *Let G be an n-vertex graph, and let $A \subseteq E(G)$ be triangle-independent. For every edge $uv \in A$, we have $d_A(u) \leq n - d_G(v)$.*

Proof. The set A cannot contain any edge uw where $w \in N_G(v)$, since then A would contain two edges of the triangle uvw. Hence $N_A(u) \subseteq V(G) - N_G(w)$. $\square$

Theorem 3.3.2. *For an n-vertex graph G with m edges,*

$$\alpha_1(G) \leq \frac{n^2}{2} - m.$$

Equality holds if and only if there exist $r_1, \ldots, r_t \geq 1$ such that $G \cong K_{r_1,r_1} \vee K_{r_2,r_2} \vee \cdots \vee K_{r_t,r_t}$.

Proof. Let $A \subseteq E(G)$ be triangle-independent, and let M be a maximal matching in A. We study the degree sum $\sum_{v \in V(G)} d_A(v)$ by splitting it into the sum $\sum_{v \in V(M)} d_A(v) + \sum_{v \notin V(M)} d_A(v)$.

For each v covered by M, let v' be its mate in M. Applying Lemma 3.3.1 to both endpoints of each edge in M, we obtain the bound

$$\sum_{v \in V(M)} d_A(v) \leq \sum_{v \in V(M)} (n - d_G(v')) = \sum_{v \in V(M)} (n - d_G(v)).$$

To bound $\sum_{v \notin V(M)} d_A(v)$, we first observe that the vertices not covered by M form an independent set in A, since any edge joining such vertices could be added to obtain a larger matching.

Now let v be any vertex not covered by M. For each edge $ww' \in M$, if $vw \in A$ then $vw' \notin E(G)$, since otherwise A contains two edges of the triangle vww'. Hence $d_A(v) \leq n - 1 - d_G(v)$, since each A-edge vw is witnessed by a pair $vw' \in \binom{V(G)}{2} - E(G)$. Summing this inequality over all uncovered v yields

$$\sum_{v \notin V(M)} d_A(v) \leq \sum_{v \notin V(M)} (n - 1 - d_G(v)) \leq \sum_{v \notin V(M)} (n - d_G(v)). \tag{3.1}$$

Combining this with the bound on $\sum_{v \in V(M)} d_A(V)$ and applying the Degree-Sum Formula for G yields

$$\sum_{v \in V(G)} d_A(v) \leq \sum_{v \in V(G)} (n - d_G(v)) = n^2 - 2m.$$

By the Degree-Sum Formula, the left side is $2\alpha_1(G)$, which yields the first claim.

Now we characterize the graphs for which equality holds. If $|A| = n^2/2 - m$, then every maximal matching in A is perfect, since M was an arbitrary maximal matching in A and equality can only hold in (3.1) if the sum is empty.

Let P_4 denote the path on four vertices. We claim that if A contains an induced subgraph isomorphic to P_4, then A contains a nonperfect maximal matching. Let $v_1, \ldots, v_4$ be the vertices of an induced copy of P_4 in A, written in order, and let M be any maximal matching containing the edges v_1v_2 and v_3v_4; we may assume that M is a perfect matching. Now $M - \{v_1v_2, v_3v_4\} + \{v_2v_3\}$ is a nonperfect maximal matching, since $v_1v_4 \notin A$.

Thus, if equality holds in the bound, then A is a triangle-free graph with a perfect matching and no induced P_4. This implies that every component of A is a balanced complete bipartite graph.

Next we claim that if u and v are vertices in different components of A, then $uv \in E(G)$. Suppose $uv \notin E(G)$, and let $G' = G + uv$. Now A still contains at most one edge from any triangle of G': if not, then A contains two edges of uvz, for some $z \in V(G')$. Since $uv \notin A$, this implies that $uz \in A$ and $vz \in A$, contradicting the hypothesis that u and v are in different components of A. It follows that $|A| \le n^2/2 - |E(G')| < n^2/2 - m$, contradicting the assumption that $|A| = n^2/2 - m$.

We have shown that the vertices of G can be covered by vertex-disjoint balanced complete bipartite graphs, and that if u and v are covered by different graphs, then $uv \in E(G)$. This implies that G is isomorphic to $K_{r_1,r_1} \vee \cdots \vee K_{r_t,r_t}$, as desired. $\square$

Theorem 3.3.2 suggests a possible approach to proving Conjecture 3.1.1. Observe that we have an obvious lower bound of $\tau_1(G) \ge m - n^2/4$, since Mantel's Theorem says that any triangle-free subgraph of G has at most $n^2/4$ edges.

Let x and y be defined by

$$\alpha_1(G) = \frac{n^2}{2} - m - x,$$
$$\tau_1(G) = m - \frac{n^2}{4} + y.$$

The bounds $\alpha_1(G) \le n^2/2 - m$ and $\tau_1(G) \ge m - n^2/4$ imply that $x, y \ge 0$. Now $\alpha_1(G) + \tau_1(G) = n^2/4 + (y - x)$, so for Conjecture 3.1.1 to hold for G, we need $y \le x$. As such, Conjecture 3.1.1 can be rephrased in terms of these bounds:

Conjecture 3.3.3. *For all graphs G and all $x \ge 0$, if $\alpha_1(G) \ge n^2/2 - m - x$, then $\tau_1(G) \le m - n^2/4 + x$.*

The characterization of graphs with $\alpha_1(G) = n^2/2 - m$ is the $k = 0$ case of Conjecture 3.3.3.

Chapter 4

On $(4m : 2m)$-Choosable Graphs

This chapter is based on joint work with Jixian Meng and Xuding Zhu.

4.1 Introduction

List coloring of graphs was introduced independently in the 1970s by Vizing [38] and by Erdős, Rubin, and Taylor [15] and has been studied extensively in the literature [35]. List coloring generalizes classical graph coloring. A *list assignment* is a function L that assigns to each vertex v a set of permissible colors $L(v)$. A graph G is k*-choosable* if, for any list assignment L with $|L(v)| \leq k$ for all v, there exists a proper coloring ϕ such that $\phi(v) \in L(v)$ for all vertices v. The *choice number* of a graph G is the smallest integer k such that G is k-choosable.

Another generalization of classical graph coloring is *fractional coloring*. For any positive integer b, a b*-tuple coloring* of a graph G is an assignment of b distinct colors to each vertex of G such that adjacent vertices receive disjoint sets of colors. If ϕ is a b-tuple coloring of G such that $\phi(v) \subseteq \{1, \ldots, a\}$ for all $v \in V(G)$, then ϕ is called a b*-tuple* a*-coloring of* G. The *fractional chromatic number* $\chi_f(G)$ is defined by

$$\chi_f(G) = \inf\left\{\frac{a}{b} : \ G \text{ has a } b\text{-tuple } a\text{-coloring}\right\} \tag{4.1}$$

The common generalization of list coloring and fractional coloring is *fractional list coloring*, also introduced by Erdős, Rubin, and Taylor [15]. For any list assignment L, an $(L : b)$*-coloring* of G is a b-tuple coloring ϕ such that $\phi(v) \subseteq L(v)$ for all vertices v. A graph G is $(a : b)$*-choosable* if G has an $(L : b)$-coloring for every list assignment L such that $|L(v)| \geq a$ for all v. The *fractional choice number* $\mathrm{ch}_f(G)$ is defined by

$$\mathrm{ch}_f(G) = \inf\left\{\frac{a}{b} : \ G \text{ is } (a : b)\text{-choosable}\right\}. \tag{4.2}$$

It is well known that the difference $\mathrm{ch}(G) - \chi(G)$ can be arbitrarily large; in particular, there exist bipartite graphs with arbitrarily high choice number. On the other hand, Alon, Tuza, and Voigt [2] showed that

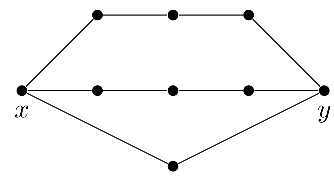


Figure 4.1: The graph $\Theta_{2,4,4}$.

$\mathrm{ch}_f(G) = \chi_f(G)$ for every graph G and that the infima in Equations (4.1) and (4.2) are attained by some pair (a, b). In particular, for any bipartite graph G, there exists an integer m such that G is $(2m : m)$-choosable.

A class of graphs called the *generalized theta graphs* play an important role in the characterization of the 2-choosable graphs. When $a_1, \ldots, a_\ell$ are nonnegative integers, the theta graph $\Theta_{a_1,\ldots,a_\ell}$ is obtained by starting with two vertices x and y and joining these vertices with t internally disjoint paths, the ith path having a_i edges. The graphs $\Theta_{2,4,4}$ is shown in Figure 4.1.

We also define one more class of graphs: a *bicycle* is a graph consisting of two even cycles that either share a single vertex, or are vertex-disjoint and joined by a path. These graphs appear in the characterization of the 2-choosable graphs. While previous authors treated these as two distinct classes (according to whether or not the cycles are vertex-disjoint), our analysis of both types is the same, so we treat them as a single class.

Voigt [39] used Rubin's characterization (see [15]) of the 2-choosable graphs to prove the following characterization of 3-choosable-critical graphs:

Theorem 4.1.1 (Voigt [39])**.** *A graph G is 3-choosable-critical if and only if it is one of the following:*

(a) a bicycle,

(b) a $\Theta_{2r,2s,2t}$-graph or $\Theta_{2r-1,2s-1,2t-1}$-graph with with $r \geq 1$ and $s, t > 1$,

(c) a $\Theta_{2,2,2,2t}$-graph with $t \geq 1$,

(d) an odd cycle.

Confirming a special case of a conjecture in [15], Tuza and Voigt [36] proved that 2-choosable (i.e., $(2\!:\!1)$-choosable) graphs are $(2m : m)$-choosable for any positive integer m. On the other hand, Voigt [39] proved that if m is an odd integer, then the 2-choosable graphs are the only $(2m\!:\!m)$-choosable graphs. For even m, the situation is much more complicated. Voigt conjectured that all bipartite 3-choosable-critical graphs are $(4 : 2)$-choosable. However, Voigt's conjecture is false. In this chapter, we characterize the $(4 : 2)$-choosable 3-choosable-critical graphs:

Theorem 4.1.2. *A 3-choosable-critical graph G is $(4:2)$-choosable if and only if it is one of the following:*

(a) a bicycle,

(b) a $\Theta_{2,2s,2t}$-graph or $\Theta_{1,2s-1,2t-1}$-graph with $s, t > 1$,

(c) $\Theta_{2,2,2,2}$.

In particular, the graphs $\Theta_{2r,2s,2t}$ and $\Theta_{2r-1,2s-1,2t-1}$ fail to be $(4:2)$-choosable when $\min\{r, s, t\} > 1$, and $\Theta_{2,2,2,2t}$ fails to be $(4:2)$-choosable when $t > 1$. This result is somewhat counterintuitive, because it says that small theta graphs are $(4:2)$-choosable while large ones are not; however, small theta graphs have higher average degree than large theta graphs, which suggests that small theta graphs should be harder to color.

The chapter is structured as follows. In Section 4.2 we introduce the main lemmas and definitions used to prove that the graphs listed in Theorem 4.1.2 are $(4:2)$-choosable.

In Section 4.3 we determine which theta-graphs are $(4:2)$-choosable. In Section 4.4 we reuse these results to show that all bicycles are $(4:2)$-choosable. Tuza and Voigt [37] proved that $K_{2,4}$ is $(4m:2m)$-choosable for all m, so this completes the sufficiency proof of Theorem 4.1.2.

In Section 4.5, we obtain a weaker version of Voigt's Conjecture: using a result of Alon, Tuza, and Voigt [2], we show that there exists some (very large) constant k such that every bipartite 3-choosable-critical graph is $(4mk:2mk)$-choosable for all m. In Section 4.6, we close with a conjectured characterization of the $(4:2)$-choosable graphs: we conjecture that they are precisely the 3-choosable-critical graphs listed in Theorem 4.1.2 together with four families of "exceptional graphs", each having cycle rank 3.

4.2 Paths and Damage

Let P be an n-vertex path with vertices $v_1, \ldots, v_n$ in this order, and let L be a list assignment on P such that $|L(v_1)| = |L(v_n)| \geq 2m$ and $|L(v_i)| = 4m$ for $i \in \{2, \ldots, n-1\}$. We define sets X_i and a number $S_L(P)$ based on the list assignment:

Definition 4.2.1. When P and L are as above, we define sets $X_1, \ldots, X_n$ by the recurrence

$$X_1 = L(v_1),$$

$$X_i = L(v_i) - X_{i-1}, \quad \text{for } i \in \{2, \ldots, n\}.$$

We define $S_L(P)$ by

$$S_L(P) = \sum_{i=1}^{n} |X_i| .$$

Lemma 4.2.2. *Under the above hypotheses, P is $(L:2m)$-colorable if and only if $S_L(P) \geq 2mn$.*

Proof. We use induction on n. The claim is trivial for $n = 1$, so assume that $n \geq 2$. Let $P' = P - v_n$, and observe that if $X'_1, \dots X'_{n-1}$ are computed as above for P', then $X'_i = X_i$ for all $i \in \{1, \dots, n-1\}$.

First suppose that P has some $(L:2m)$-coloring ϕ; we show that $S_L(P) \geq 2mn$. Let L^* be the restriction of L to P', except that $L^*(v_{n-1}) = L(v_{n-1}) - \phi(v_n)$, and let $X^*_1, \dots X^*_n$ be computed for L^*. Since ϕ is an $(L^* : 2m)$-coloring of P', the induction hypothesis implies that

$$\sum_{i=1}^{n-1} |X^*_i| = S_{L^*}(P') \geq 2(n-1)m.$$

It is easy to verify that $X_i = X^*_i$ for $i = 1, 2, \dots, n-2$, and $|X_{n-1}| + |X_n| \geq \left|X^*_{n-1}\right| + |\phi(v_n)| \geq \left|X^*_{n-1}\right| + 2m$. Hence $S_L(P) \geq 2nm$.

Now suppose $S_L(P) \geq 2nm$; we shall prove that P is $(L:2m)$-colorable. We consider two cases: either $|X_n| \geq 2m$, or $|X_n| \leq 2m$. Let $P' = P - v_n$.

Case 1: $|X_n| \leq 2m$. Let $\phi(v_n)$ be any $2m$-subset of $L(v_n)$ containing X_n, and let L^* be the restriction of L to P', except that $L^*(v_{n-1}) = L(v_{n-1}) - \phi(v_n)$. Since $S_L(P) \geq 2mn$, and since L^* loses at most $2m - |X_n|$ colors from X_{i-1}, we have

$$S_{L^*}(P') \geq S_L(P) - |X_n| - (2m - |X_n|) \geq S_L(P) - 2m \geq 2m(n-1).$$

By the induction hypothesis, P' has an $(L^* : 2m)$-coloring, which extends to an $(L:2m)$-coloring of P by assigning the color set $\phi(v_n)$ to v_n.

Case 2: $|X_n| > 2m$. Since $|L(v_i)| = 4m$ for $i \in \{2, \dots, n-1\}$, any $(L:2m)$-coloring of v_1 can be greedily extended to an $(L:2m)$-coloring of P'; in particular, P' is $(L:2m)$-choosable. By the induction hypothesis, $S_L(P') \geq 2(n-1)m$.

Let $\phi(v_n)$ be any $2m$-subset of X_n, and let L^* be the restriction of L to P', except that $L^*(v_{n-1}) = L(v_{n-1}) - \phi(v_n)$. Since $\phi(v_n) \subseteq X_n$ and since X_{n-1} is disjoint from X_n, we have

$$S_{L^*}(P') = S_L(P') \geq 2(n-1)m.$$

Hence, by the induction hypothesis, P' has an $(L^* : 2m)$-coloring, which extends to an $(L:2m)$-coloring of

P by assigning the color set $\phi(v_n)$ to v_n. $\square$

Our typical strategy for showing that a graph G is $(4m : 2m)$-choosable is as follows: identify a set of vertices X such that $G - X$ is a linear forest (disjoint union of paths), and find a precoloring of X such that each path P in $G - X$ satisfies $S_{L^*}(P) \geq 2\,|V(P)|$, where L^* is obtained from L by removing from each vertex of $G - X$ the colors used on its neighbors in X. Lemma 4.2.2 then guarantees that we can extend the precoloring of X to the rest of the graph, as desired. The components of $G - X$ are called the *internal paths* of G.

In order to carry out this strategy, we need to know how $S_L(P)$ changes when colors are removed from the endpoints of P. Before stating the results, we introduce more notation. When we consider a set of colors in the context of coloring some vertex with that set (or removing that set from a list of colors, modeling the use of that set on a neighbor), we write that set using lowercase letters like p and q, emphasizing that when a set is used this way, it is analogous to the use of a single color in classical coloring problems.

Definition 4.2.3. If L is a list assignment on P_n and p and q are sets of colors, we define the *reduced assignment relative to* (p, q), written $L \ominus (p, q)$, to be the list assignment obtained from L by deleting all colors in p from $L(v_1)$, all colors in q from $L(v_n)$, and leaving all other lists alone.

Definition 4.2.4. Let L be a list assignment on a path P. Define $A = \bigcap_{x \in V(P)} L(x)$. For $c \in L(v_1) - A$, let $f(c) = \min\{i\colon\ c \notin L_i\}$. We define $\hat{X}_1$ by

$$\hat{X}_1 = \{c \in L(v_1) - A\colon\ f(c) \text{ is even}\}.$$

Finally, let $\hat{X}_n = X_n - A$, where $X_1, \ldots, X_n$ are as in Definition 4.2.1.

Despite the apparent asymmetry in this definition, when n is odd the sets $\hat{X}_1$ and $\hat{X}_n$ are symmetric in the following sense: if we let $Y_n = L(v_n)$ and $Y_i = L(v_i) - Y_{i+1}$ for $1 \leq i \leq n-1$, then $Y_1 - A = \hat{X}_1$.

Our next lemma applies only to paths with an odd number of vertices. While this may seem restrictive, these are the only paths that need to be explicitly dealt with in our proof.

Lemma 4.2.5. *Let L be a list assignment on an n-vertex path P, where n is odd. For any sets p and q of colors, we have*

$$S_{L \ominus (p,q)}(P) = S_L(P) - \left(\left| (A \cup \hat{X}_1) \cap p \right| + \left| (A \cup \hat{X}_n) \cap q \right| - |A \cap p \cap q| \right).$$

Proof. We can rewrite $S_L(P)$ as

$$\sum_{c \in \bigcup L(v_i)} \sum_{i=1}^{n} |\{i \in [n] : c \in X_i\}| .$$

The set $\{i \in [n] : c \in X_i\}$ depends only on which sets $L(v_i)$ contain the color c: when $d \neq c$, adding or deleting d from the lists does not alter which sets X_i contain the color c. Thus, we may think of moving from $S_L(P)$ to $S_{L\ominus(p,q)}(P)$ as a gradual process where we delete one color at a time, allowing us to consider the effect of removing just that color, independent of the effect the other colors had. We let S denote the "current value" of the sum as we move from $S_L(P)$ to $S_{L\ominus(p,q)}(P)$, and determine the change in S caused by deleting just one color c.

First we consider deleting some color $c \in q$ from $L(v_n)$. Clearly, if $c \notin X_n$ then deleting the color c from $L(v_n)$ has no effect on S, since it does not change any X_i. On the other hand, if $c \in X_n = A \cup \hat{X}_n$, then deleting the color c from $L(v_n)$ decreases S by exactly 1.

Next we consider deleting a color $c \in p$ from $L(v_1)$, assuming that it has already been deleted from $L(v_n)$ if $c \in p \cap q$. Here, unlike with $L(v_n)$, the changes in X_1 can "ripple" through later X_i. If $c \notin X_1 = L(v_1)$, then deleting c from $L(v_1)$ does not change any X_i and hence does not change S.

Now suppose $c \in X_1 - A$. Deleting c from $L(v_1)$ also removes c from X_1. However, if $f(c) \neq 2$, then $c \in L(v_2)$, so we add c to X_2. If $c \in X_3$ initially, then adding c to X_2 deletes c from X_3. If also $c \in L(v_4) - X_4$, then deleting c from X_3 adds c to X_4. The process of alternately deleting and adding c continues until we reach $v_{f(c)}$, which does not have c in its list, so $X_{f(c)}$ does not change. In total, we delete c from the sets $X_1, X_3, \ldots, X_{f(c)-2}$ and add c to the sets $X_2, X_4, \ldots, X_{f(c)-1}$. If $f(c)$ is odd, there is no net change in S. If $f(c)$ is even, then S has decreased by 1.

Finally, suppose $c \in X_1 \cap A$. Deleting c from $L(v_1)$ causes the same ripple process described above, terminating when we try to delete c from X_n (since n is odd). If $c \notin q$, then this causes S to decrease by 1, as before. However, if $c \in q$, then we have *already* deleted c from X_n, so in this step we actually have no net change in the total usage of c (effectively, deleting c from X_n caused c to leave A and caused $f(c)$ to become odd). Thus, when $c \in p \cap q \cap A$, deleting c from both endpoints of L decreases S by exactly 1, but such colors are double-counted in the sum $\left|(A \cup \hat{X}_1) \cap p\right| + \left|(A \cup \hat{X}_n) \cap q\right|$. The final term $|A \cap p \cap q|$ corrects for this overcount. $\square$

Together, Lemma 4.2.2 and Lemma 4.2.5 allow us to ignore the details of the list assignment and focus on the sets $\hat{X}_1, \hat{X}_n, A$, as described below:

Definition 4.2.6. For sets p and q of colors, the *damage* of (p, q) with respect to L and P is written

$\mathrm{dam}_{L,P}(p,q)$ and defined by

$$\mathrm{dam}_{L,P}(p,q) = S_L(P) - S_{L\ominus(p,q)}(P).$$

When $\mathrm{dam}_{L,P}(p,q) = k$ and when L, P are understood, we also say that the pair (p,q) *does damage* k.

We note briefly how this definition fits into our strategy for coloring theta-graphs, say to show that $\Theta_{2,4,4}$ is $(4:2)$-choosable. Let L be any $(4:2)$-assignment on $\Theta_{2,4,4}$, and let x and y be the vertices of degree 3. We seek a partial L-coloring of x and y that can be extended to the remaining vertices. Indexing each internal path so that v_1 is adjacent to x and v_n is adjacent to y, we see that precoloring x with the set p and precoloring y with the set q causes damage $\mathrm{dam}_{L,P}(p,q)$ to each internal path P. By Lemma 4.2.2, we can complete the coloring on P if and only if $\mathrm{dam}_{L,P}(p,q) \le S_L(P) - 2m\,|V(P)|$. Thus, roughly speaking, our goal is to choose a precoloring that causes only a small amount of damage to each internal path, so that we can then complete the precoloring on each internal path. This discussion is summarized in the following lemma.

Lemma 4.2.7. *Let G be a graph, and let $X \subseteq V(G)$ be a set of vertices such that every component of $G - X$ is a path with an odd number of vertices. The graph G is $(L:2m)$-colorable if and only if $G[X]$ has an $(L:2m)$-coloring ϕ such that for every component P in $G - X$, the following conditions hold:*

(i) $|L(v_1) \cap \phi(N_X(v_1))| \le 2m$,

(ii) $|L(v_n) \cap \phi(N_X(v_n))| \le 2m$, *and*

(iii) $\mathrm{dam}_{L,P}(\phi(N_X(v_1)), \phi(N_X(v_n))) \le S_L(P) - 2mn$.

Proof. Clearly, G is $(L:2m)$-colorable if and only if $G[X]$ has an $(L:2m)$-coloring ϕ that extends to P. For each component P, we show that ϕ extends to P if and only if ϕ satisfies conditions (i)–(iii). Conditions (i) and (ii) are clearly necessary, so it suffices to show that when Conditions (i) and (ii) hold, ϕ extends to P if and only if Condition (iii) holds. This follows from Lemma 4.2.2 and Lemma 4.2.5.

Hence, we may extend ϕ to each path in $G - X$. Since these paths are separate components of $G - X$, making all of these extensions simultaneously yields an $(L:2m)$-coloring of G. $\square$

Observation 4.2.8. If v_1 and v_n also have degree 2, then Conditions (i) and (ii) are trivially satisfied by any ϕ, so we only need to check Condition (iii).

To apply Lemma 4.2.7, we need to find lower bounds for $S_L(P)$ and upper bounds for $\mathrm{dam}_{L,P}(p,q)$. We spend the rest of the section proving such bounds.

Lemma 4.2.9. *If L is a list assignment on an n-vertex path P, where n is odd and $|L(v_i)| = 4m$ for all i, then*

$$S_L(P) = 2nm - 2m + \sum_{\substack{k \text{ even} \\ k<n}} |X_{k-1} - L(v_k)| + |X_n|$$

Proof. We use induction on n. When $n = 1$, the sum is empty and $2nm - 2m = 0$, so the claim is just $S_L(P) = |X_1|$, which is clearly true. Now consider $n > 1$. Let $P' = P - \{v_{n-1}, v_n\}$ and let L' be the restriction of L to P', so that $S_L(P) = S_{L'}(P') + |X_{n-1}| + |X_n|$. Applying the induction hypothesis to P' yields

$$S_L(P) = \left(2nm - 6m + \sum_{\substack{k \text{ even} \\ k<n-2}} |X_{k-1} - L(v_k)| + |X_{n-2}| \right) + |X_{n-1}| + |X_n|$$

Observe that

$$\begin{aligned}
|X_{n-1}| &= |L(v_{n-1}) - X_{n-2}| \\
&= |L(v_{n-1})| - |X_{n-2}| + |X_{n-2} - L(v_{n-1})| \\
&= 4m - |X_{n-2}| + |X_{n-2} - L(v_{n-1})|
\end{aligned}$$

Combining these terms with the terms from $S_{L'}(P')$ gives the desired expression for $S_L(P)$. □

The following two lower bounds on $S_L(P)$ will be useful.

Lemma 4.2.10. *If L is a list assignment on an n-vertex path P, where n is odd and $|L(v_i)| = 4m$ for all i, then*

$$S_L(P) \geq 2nm - 2m + \left|\hat{X}_1\right| + \left|\hat{X}_n\right| + |A|.$$

Proof. By the definition of $\hat{X}_1$, every element of $\hat{X}_1$ appears in a set of the form $X_{k-1} - L(v_k)$ where for some even k. Thus, the claim follows from Lemma 4.2.9, since $|X_n| = \left|\hat{X}_n\right| + |A|$. □

Lemma 4.2.11. *If L is a list assignment on an n-vertex path P, where n is odd and $|L(v_i)| = 4m$ for all i, then $S_L(P) \geq 2nm + 2m$.*

Proof. This follows immediately from the definition $S_L(P) = \sum_{i=1}^{n} |X_i|$ and the observations that $|X_1| = |L(v_1)| = 4m$ and that $|X_i| + |X_{i+1}| \geq 4m$ for $i > 1$. □

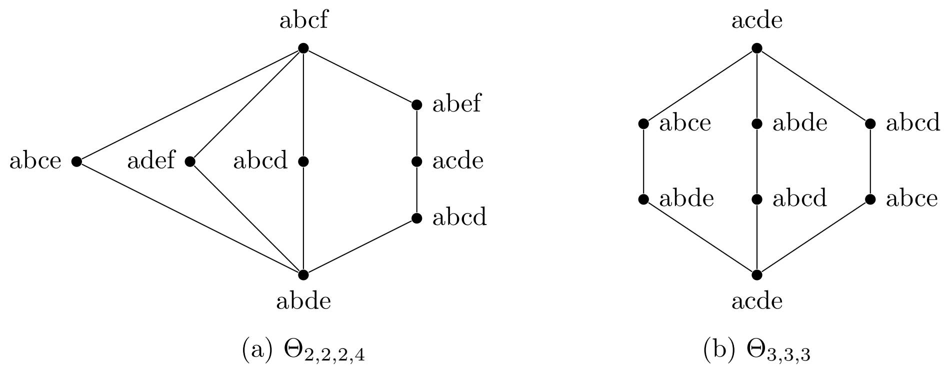


Figure 4.2: Noncolorable list assignments for $\Theta_{2,2,2,4}$ and $\Theta_{3,3,3}$.

4.3 $(4:2)$-choosability of Theta Graphs

In the remainder of this chapter, we concentrate on $(4:2)$-choosability of graphs, rather than general $(4m:2m)$-choosability. Thus, we assume that the list assignment L has $|L(v)| = 4$ for all vertices v. In this section, we prove the following lemma:

Lemma 4.3.1. *When r, s, and t are positive integers, the graph $\Theta_{2r,2s,2t}$ or $\Theta_{2r-1,2s-1,2t-1}$ is $(4:2)$-choosable if and only if $\min\{r,s,t\} = 1$. When t is a positive integer, the graph $\Theta_{2,2,2,2t}$ is $(4:2)$-choosable if and only if $t = 1$.*

Note that if not all a_i have the same parity, then the graph $\Theta_{a_1,\dots,a_\ell}$ contains an odd cycle and is therefore not $(4:2)$-choosable, so the graphs considered in Lemma 4.3.1 are the only ones that need to be considered.

The following observation, due to Voigt, allows us to better focus our attention, restricting the class of graphs we need to consider for both the necessity and sufficiency parts of the proof.

Lemma 4.3.2 (Voigt [39])**.** *Let G be a graph, let $v \in V(G)$, and let G' be obtained from G by deleting all the neighbors of v and then making v adjacent to all vertices w such that $d_G(v,w) \le 2$. If G is $(4:2)$-choosable, then G' is $(4:2)$-choosable.*

The transformation used in Lemma 4.3.2 was first used in [15], which observed that if G is 2-choosable, then G' is also 2-choosable. Voigt [39] made the stronger observation that if G is $(2m:m)$-choosable, then G' is also $(2m:m)$-choosable.

We first prove necessity in Lemma 4.3.1. That is, we argue that if $\min\{r,s,t\} \ge 2$, then $\Theta_{2r,2s,2t}$ and $\Theta_{2r-1,2s-1,2t-1}$ are not $(4:2)$-choosable, and that if $t \ge 2$, then $\Theta_{2,2,2,2t}$ is not $(4:2)$-choosable. Figure 4.2 shows noncolorable list assignments for $\Theta_{2,2,2,4}$ and $\Theta_{3,3,3}$. To show that larger theta graphs are not $(4:2)$-choosable, we again apply Lemma 4.3.2. In particular, the contrapositive of Lemma 4.3.2 states that if G' is not $(4:2)$-choosable, then G is not $(4:2)$ choosable either. Hence $\Theta_{4,4,4}$ is not $(4:2)$-choosable, since $\Theta_{3,3,3}$ is obtained from $\Theta_{4,4,4}$ by applying this reduction to a vertex of degree 3.

Likewise, $\Theta_{2,2,2,2t+2}$ is obtained from $\Theta_{2,2,2,2t}$ by applying this reduction to a vertex of degree 2; hence, since $\Theta_{2,2,2,4}$ is not $(4:2)$-choosable, it follows by induction on t that when $t \geq 2$, the graph $\Theta_{2,2,2,2t}$ is not $(4:2)$-choosable. Similarly, since $\Theta_{3,3,3}$ is not $(4:2)$-choosable, no graph of the form $\Theta_{2r-1,2s-1,2t-1}$ for $r, s, t \geq 2$ is $(4:2)$-choosable, and since $\Theta_{4,4,4}$ is not $(4:2)$-choosable, no graph of the form $\Theta_{2r,2s,2t}$ for $r, s, t \geq 2$ is $(4:2)$-choosable.

Now we prove sufficiency. Lemma 4.3.2 has the following corollary, which allows us to restrict to the case where all internal paths have an even number of edges.

Corollary 4.3.3. *If $\Theta_{2,2s,2t}$ is $(4:2)$-choosable, then $\Theta_{1,2s-1,2t-1}$ is $(4:2)$-choosable.*

Proof. Applying the operation of Lemma 4.3.2 to a vertex v of degree 3 transforms $\Theta_{2,2s,2t}$ into $\Theta_{1,2s-1,2t-1}$. □

It therefore suffices to show that $\Theta_{2,2s,2t}$ is $(4:2)$-choosable for all $s, t \geq 1$. Similar techniques will allow us to deal with bicycles.

The vertices of degree 3 play a special role in the proof; we call them x and y. The idea is to show that for any list assignment L on $\Theta_{2,2s,2t}$, there is a precoloring of the vertices x and y, giving the color sets p and q to the vertices x and y respectively, such that $\text{dam}_{L,P}(p, q) \leq S_L(P) - 2\,|V(P)|$ for all internal paths P. By Lemma 4.2.7, this implies that $\Theta_{2,2s,2t}$ is $(4:2)$-choosable. We use the word *precoloring* to refer to a precoloring of just the vertices x and y, and write such a precoloring as a pair (p, q), where p and q are the color sets given to x and y respectively.

Fix a list assignment L, and let $L(x) = \{c_0, c_1, c_2, c_3\}$ and $L(y) = \{c'_0, c'_1, c'_2, c'_3\}$, where the colors are indexed so that $c'_j = c_j$ whenever $c_j \in L(x) \cap L(y)$. For a given indexing, a *couple* is a pair of colors having the form (c_i, c'_i) for some $i \in \{0, 1, 2, 3\}$. To visually distinguish couples from precolorings, we suppress parentheses and commas when writing couples, writing $c_ic'_i$ instead of (c_i, c'_i).

A *coupled precoloring* is a precoloring (p, q) such that for all $j \in \{0, 1, 2, 3\}$, $c_j \in p$ if and only if $c'_j \in q$. A *good precoloring* is a coupled precoloring (p, q) such that $\text{dam}(p, q) \leq S_L(P) - 2\,|V(P)|$ for all internal paths P. Observe that the definition of coupled precoloring depends on the indexing of the colors in $L(x)$ and $L(y)$. Fixing some indexing, we first try to find a good precoloring. We show that a good precoloring exists unless L has a very specific form. Then we address this form as a special case.

Lemma 4.2.5 implies that if (p, q) is a coupled precoloring, then

$$\operatorname{dam}(p, q) = \sum_{\substack{c_j \in p \\ c_j' \in q}} \operatorname{dam}(\{c_j\}, \{c_j'\}).$$

In other words, when (p, q) is a coupled precoloring, we can simply calculate the damage of each couple independently, and add them together to obtain $\operatorname{dam}(p, q)$. Moreover, $\operatorname{dam}_{L,P}(\{c_i\}, \{c_i'\}) \in \{0, 1, 2\}$ for each i. We say that:

- The couple $c_j c_j'$ is *heavy* for the internal path P if $\operatorname{dam}_{L,P}(\{c_j\}, \{c_j'\}) = 2$;
- The couple $c_j c_j'$ is *light* for the internal path P if $\operatorname{dam}_{L,P}(\{c_j\}, \{c_j'\}) = 1$;
- The couple $c_j c_j'$ is *safe* for the internal path P if $\operatorname{dam}_{L,P}(\{c_j\}, \{c_j'\}) = 0$.

Definition 4.3.4. We say that an internal path P *blocks* a precoloring (p, q) if $\operatorname{dam}_{L,P}(p, q) > S_L(P) - 2\,|V(P)|$, i.e., if we cannot extend the precoloring to all of P.

Observe that if $c_j c_j'$ is heavy for P, then $c_j \in \hat{X}_1$ and $c_j' \in \hat{X}_n$; if $c_j c_j'$ is light for P, then either $c_j = c_j'$ and $c_j \in A$, or $\left|\{c_j\} \cap \hat{X}_1\right| + \left|\{c_j'\} \cap \hat{X}_n\right| = 1$.

Now we count how many coupled precolorings are blocked by each internal path. This lemma actually holds for any theta graph, not necessarily a theta graph with three paths.

Lemma 4.3.5. *If $a_1, \ldots, a_\ell$ are positive even integers, then in the theta graph $\Theta_{a_1,\ldots,a_\ell}$, each internal path P blocks at most two coupled precolorings. If P blocks two coupled precolorings, then $S_L(P) = 2\,|V(P)| + 2$ and P has one heavy couple and two light couples.*

Proof. Let $n = |V(P)|$. By the case $m = 1$ of Lemma 4.2.11, $S_L(P) \geq 2n + 2$. If $S_L(P) \geq 2n + 4$, then P does not block any coupled precolorings, since $\operatorname{dam}(p, q) \leq 4$ for any precoloring (p, q). Hence it suffices to consider $S_L(P) \in \{2n + 2, 2n + 3\}$.

In both cases, P has at most two heavy couples: if $c_j c_j'$ is a heavy couple, then $c_j \in \hat{X}_1$ and $c_j' \in \hat{X}_n$, so the $m = 1$ case of Lemma 4.2.10 implies that if P has two heavy couples, then $S_L(P) \geq 2n + 4$.

If $S_L(P) = 2n + 3$, then P only blocks the coupled precoloring (p, q) if $\operatorname{dam}(p, q) = 4$, i.e., if both couples used in (p, q) are heavy. Since P has at most two heavy couples, this implies that P blocks at most one coupled precoloring.

If $S_L(P) = 2n + 2$, then P blocks the coupled precoloring (p, q) if and only if $\operatorname{dam}(p, q) \geq 3$, i.e., if one of the couples in (p, q) is heavy and the other is not safe. Lemma 4.2.10 implies that if P has two heavy couples, then P has no light couple, since if $c_jc'_j$ is light, then either $c_j \in \hat{X}_1 \cup A$ or $c'_j \in \hat{X}_n$. Likewise, if P has one heavy couple, then P has at most two light couples. The desired conclusion follows. $\square$

Now we specialize to the case $\Theta_{2r,2s,2t}$. While we are only proving $(4:2)$-choosability for graphs of the form $\Theta_{2,2s,2t}$, this corollary also gives information about the form of non-colorable list assignments when $r \geq 2$.

Corollary 4.3.6. *In the theta graph $\Theta_{2r,2s,2t}$, if L has no good precoloring, then every coupled precoloring (p, q) is blocked by exactly one internal path. In particular, each couple $c_jc'_j$ is heavy for at most one internal path P.*

Proof. There are six coupled precolorings, and each of the three internal paths blocks at most two of them; this proves the first part. If the couple $c_jc'_j$ is heavy for two different internal paths P and Q, then since P and Q each have two light couples, there is some couple $c_kc'_k$ that is light for both P and Q. Now the precoloring $(\{c_j, c_k\}, \{c'_j, c'_k\})$ is blocked by both P and Q, contradicting the first part of the corollary. $\square$

We now must handle the case where L has no good precoloring. Up until now, we have not used the fact that $r = 1$; that assumption is only used in this case. First we refine our notation. Let P^0, P^1, P^2 be the internal paths of $\Theta_{2,2s,2t}$, with $\left|V(P^0)\right| = 1$. By Corollary 4.3.6, we may reindex $L(x)$ and $L(y)$ so that for all $j \in \{0, 1, 2\}$, $c_jc'_j$ is heavy for P^j. (By simultaneously permuting the labels in $L(x)$ and $L(y)$, this maintains the original property that $c'_j = c_j$ whenever $c_j \in L(x) \cap L(y)$.) With this new notation, we have the following further consequence of Corollary 4.3.6:

Corollary 4.3.7. *If L has no good precoloring, then $c_3c'_3$ is light for all internal paths P^i, and one of the two following situations must hold:*

(a) $c_1c'_1$ is light for P^0, $c_2c'_2$ is light for P^1, and $c_0c'_0$ is light for P^2, or

(b) $c_2c'_2$ is light for P^0, $c_0c'_0$ is light for P^1, and $c_1c'_1$ is light for P^2.

Proof. We know that if L has no good precoloring, then every coupled precoloring is blocked by exactly one internal path, and that each internal path blocks exactly two coupled precolorings. By Lemma 4.3.5, each internal path has exactly one heavy couple and exactly two light couples. Since each couple $c_ic'_i$ is heavy for P^i, it follows that $c_3c'_3$ is not heavy for any P^i. If $c_3c'_3$ is safe for some P^i, then the coupled precoloring $(\{c_i, c_3\}, \{c'i, c'_3\})$ is not blocked: when $j \neq i$, neither $c_ic'_i$ nor $c_3c'_3$ is heavy for P^j, so $\operatorname{dam}_{L,P^j}(\{c_i, c_3\}, \{c'_ic'_3\}) \leq 2$ for all j. Thus, $c_3c'_3$ is light for all internal paths.

Next, we claim every couple c_ic_i' for $i \in \{0,1,2\}$ is light for some path. If c_ic_i' is not light for any path, then let c_jc_j' be a couple that is safe for P^i; the coupled precoloring $(\{c_i, c_j\}, \{c_i', c_j'\})$ is not blocked by any pair, since c_i is not light for any internal path and is only heavy for P^i.

Thus, if for $i \in \{0,1,2\}$ we define $\pi(i)$ to be the unique index $j \in \{0,1,2\}$ such that c_jc_j' is light for P^i, we have shown that the map π is a permutation of $\{0,1,2\}$. Since c_ic_i' is heavy for P^i, the permutation π has no fixed points.

Now we claim that there cannot be a pair (i,j) for distinct $i,j \in \{0,1,2\}$ such that c_ic_i' is light for P^j and c_jc_j' is light for P^i. If such a pair exists, then the coupled precoloring $(\{c_i, c_j\}, \{c_i', c_j'\})$ is blocked by both P^i and P^j, contradicting the fact that every precoloring is blocked by exactly one internal path. This means that the permutation π contains no 2-cycles; thus, since π has no fixed points, π is a 3-cycle. If $\pi(0) = 1$, then $\pi(1) = 2$ and $\pi(2) = 1$, so we are in the first situation. If $\pi(0) = 2$, then $\pi(2) = 1$ and $\pi(1) = 0$, so we are in the second situation. □

The two situations in Corollary 4.3.7 are symmetric in P^1 and P^2. Since the desired result is also symmetric in P^1 and P^2, it suffices to handle situation (a).

Since $|V(P^0)| = 1$, we have $\hat{X}_1 = \hat{X}_n = \emptyset$ for P^0. Hence, since c_0c_0' is heavy for P_0, we must have $c_0 \neq c_0'$. Hence $c_0 \notin L(x) \cap L(y)$ and $c_0' \notin L(x) \cap L(y)$.

Now consider P^2. Since $c_0 \neq c_0'$ and c_0c_0' is light for P^2, we have either $c_0 \notin \hat{X}_1$ or $c_0' \notin \hat{X}_n$. If $c_0 \notin \hat{X}_1$, then let $p = \{c_0, c_3\}$ and let $q = \{c_2', c_3'\}$.

We check that (p,q) does damage at most 2 to each internal path. Since c_4c_4' is light for all internal paths, the couple c_3c_3' does at most 1 damage to each internal path. Taking $c_0 \in p$ does damage 1 to P^0 and no damage to P^1 and P^2, since c_0c_0' is safe for P^1 and $c_0 \notin \hat{X}_1$. Taking $c_2' \in q$ does damage 1 to P^1 and P^2 and no damage to P^0, since c_2c_2' is safe for P^0. Hence we have done damage at most 2 to each internal path, as desired.

The case $c_0' \notin \hat{X}_n$ is similar; here, we take $p = \{c_2, c_3\}$ and $q = \{c_0', c_3'\}$.

4.4 Bicycles are $(4:2)$-choosable

In this section, we show that all bicycles are $(4:2)$-choosable. In fact, one can show that bicycles are $(4m:2m)$-choosable for all m; here, we prove only the $(4:2)$-choosability case, which allows us to reuse some tools from the previous section.

Definition 4.4.1. Let P be a path with an odd number of vertices, let L be a list assignment on P, and let W be a set of four colors. A *bad W-set* for P is a set $p \subseteq W$ of two colors such that $\mathrm{dam}(p,p) > S_L(P) - 2|V(P)|$.

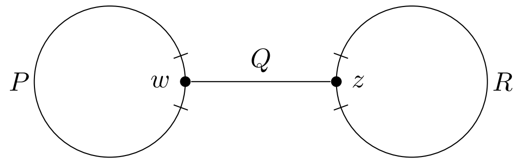


Figure 4.3: Decomposing G into P, Q, R.

Lemma 4.4.2. *If P is a path with an odd number of vertices and W is any set of four colors, then P has at most two bad W-sets.*

Proof. Consider the graph H obtained by adding new vertices x and y on the ends of P, with $L(x) = L(y) = W$. Considering P as an internal path in H (as in Section 4.3), we see that p is a bad set for P if and only if P blocks the coupled precoloring (p, p). By Lemma 4.3.5, it follows that P has at most two bad sets. $\square$

Lemma 4.4.3. *Let P be a path with endpoints w and z. For every list assignment L on P, there is an injective function $h : \binom{L(w)}{2} \to \binom{L(z)}{2}$ such that for all $p \in \binom{L(w)}{2}$, the precoloring $\phi(w) = p$, $\phi(z) = h(p)$ extends to all of P.*

Proof. We use induction on $|V(P)|$. When $|V(P)| = 1$ or $|V(P)| = 2$, the claim clearly holds: when $|V(P)| = 1$ we may take h to be the identity function, and when $|V(P)| = 2$ it suffices that $p \cap h(p) = \emptyset$ for all p; such an h is easy to construct.

Hence we may assume that $|V(P)| > 2$ and the claim holds for smaller P. Let z' be the unique neighbor of z. We split P into the z, z'-subpath Q_1 and the z', z-subpath Q_2, overlapping only at v'. Let h_1 and h_2 be the functions for Q_1 and Q_2 respectively, as guaranteed by the induction hypothesis. Composing h_2 and h_1, we see that $h_2 \circ h_1$ has the desired properties. $\square$

We handle "two cycles sharing a vertex" as a special case of "two cycles joined by a path", considering the shared vertex as a path on one vertex.

Corollary 4.4.4. *If G is a bicycle, then G is $(4:2)$-choosable.*

Proof. Let C and D be the cycles in G, and let $w \in V(C)$ and $z \in V(D)$ be the endpoints of the path joining C and D. Let $P = C - w$, let $R = D - z$, and let Q be the path joining w and z, so that P, Q, R are disjoint paths with $V(P) \cup V(Q) \cup V(R) = V(G)$. The situation is illustrated in Figure 4.3. By Lemma 4.4.2, the path P has at most two bad $L(w)$-sets, and the path R has at most two bad $L(z)$-sets. Let $h : \binom{L(w)}{2} \to \binom{L(z)}{2}$ be the injection guaranteed by Lemma 4.4.3. Since there are six ways to choose a set $p \in \binom{L(w)}{2}$, we see that there is some p such that p is not bad for P and $h(p)$ is not bad for Q. It follows that we may extend the precoloring $\phi(w) = p$, $\phi(z) = h(p)$ to all of P, Q, and R. $\square$

Tuza and Voigt [37] proved that $K_{2,4}$ is $(4m : 2m)$-choosable for all m, so this completes the positive direction of Theorem 4.1.2.

4.5 A Conjecture of Voigt

Voigt [39] conjectured that every bipartite 3-choosable-critical graph is $(4m : 2m)$-choosable for all m. We have seen that this conjecture fails for $m = 1$: there exist non-$(4 : 2)$-choosable 3-choosable-critical graphs. However, one can prove the following weaker version of Voigt's conjecture:

Theorem 4.5.1. *There exists a constant k such that every bipartite 3-choosable-critical graph is $(4mk{:}2mk)$-choosable for all m.*

Our proof is based on the following theorem of Alon, Tuza, and Voigt [2].

Theorem 4.5.2 (Alon–Tuza–Voigt [2])**.** *For every integer n there exists a number $f(n) \le (n+1)^{2n+2}$ such that the following holds: For every graph G with n vertices and with fractional chromatic number χ^*, and for every integer M which is divisible by all integers from 1 to $f(n)$, G is $(M : M/\chi^*)$-choosable.*

Lemma 4.2.10 and Lemma 4.2.11 suggest that when n is odd, the "worst case" tuples $(A, \hat{X}_1, \hat{X}_n)$ are those satisfying $|A| + \left|\hat{X}_1\right| + \left|\hat{X}_n\right| = 4m$. The following lemma shows that any such sets can be "realized" on a path of length 3:

Lemma 4.5.3. *Let B, Y, Z be sets such that $B \cap Y = \emptyset$, $B \cap Z = \emptyset$, and $|B| + |Y| + |Z| = 4m$. There exists a list assignment L on P_3 such that:*

(i) $|L(v)| = 4m$ *for all* $v \in V(P_3)$*, and*

(ii) $(A, \hat{X}_1, \hat{X}_3) = (B, Y, Z)$*, and*

(iii) $S_L(P_3) = 8m$*.*

Proof. Let J_1 and J_2 be sets disjoint from each other and disjoint from $B \cup Y \cup Z$ such that

$$|J_1| = 4m - |B| - |Y|,$$

$$|J_2| = 4m - |B| - |Z|.$$

Observe that

$$|B| + |J_1| + |J_2| = 8m - |B| - |Y| - |Z| = 4m.$$

Consider the following list assignment on P_3:

$$L(v_1) = B \cup Y \cup J_1,$$
$$L(v_2) = B \cup J_1 \cup J_2,$$
$$L(v_3) = B \cup Z \cup J_2.$$

We verify that L has the desired properties:

(i) Since $B \cap Y = \emptyset$ and $|J_1| = 4m - |B| - |Y|$, we have $|B \cup Y \cup J_1| = 4m$. Similarly, $|B \cup Z \cup J_2| = 4m$. Finally,

$$\begin{aligned} |B \cup J_1 \cup J_2| &= |B| + (4m - |B| - |Y|) + (4m - |B| - |Z|) \\ &= 8m - |B| - |Y| - |Z| \\ &= 4m. \end{aligned}$$

(ii) Since the sets J_1, J_2, and $B \cup Y \cup Z$ are pairwise disjoint, we have $L(v_1) \cap L(v_2) \cap L(v_3) = B$, so $A = B$. Computing the sets X_i, we obtain

$$X_1 = B \cup Y \cup J_1,$$
$$X_2 = J_2,$$
$$X_3 = B \cup Z.$$

For each $c \in Y$, we have $f(c) = 2$, while for $c \in J_1$, we have $f(c) = 3$. Thus, $\hat{X}_1 = Y$. Similarly, $\hat{X}_3 = Z$.

(iii) From the computation of X_i in the previous part, we obtain

$$\begin{aligned} S_L(P_3) &= 2\,|B| + |Y| + |Z| + |J_1| + |J_2| \\ &= 2\,|B| + |Y| + |Z| + (4m - |B| - |Y|) + (4m - |B| - |Z|) \\ &= 8m. \end{aligned}$$

□

Lemma 4.5.3 allows us to obtain a partial converse of Lemma 4.3.2, subject to certain restrictions on the choice of the vertex v.

Lemma 4.5.4. *Let G be a graph containing a path P on five vertices which all have degree 2 in G, and let G' be the graph obtained by applying the operation of Lemma 4.3.2 to the middle vertex of P. The graph G is $(4m:2m)$-choosable if and only if the graph G' is $(4m:2m)$-choosable.*

Proof. By Lemma 4.3.2, it suffices to show that if G' is $(4m:2m)$-choosable, then G is $(4m:2m)$-choosable.

Let L be any list assignment for G such that $|L(v)| = 4m$ for all $v \in V(G)$, and let $A, \hat{X}_1, \hat{X}_5$ be computed relative to P. We will define sets B, Y, Z based on $A, \hat{X}_1, \hat{X}_5$ and apply Lemma 4.5.3 to obtain a list assignment L' on the shorter path P'. The definition is slightly different depending on whether $|A| + \left|\hat{X}_1\right| + \left|\hat{X}_5\right| \leq 4m$: we either arbitrarily add elements or arbitrarily remove elements in order to reach the desired sum.

- When $|A| + \left|\hat{X}_1\right| + \left|\hat{X}_5\right| \leq 4m$, let B, Y, Z be arbitrary supersets of $A, \hat{X}_1, \hat{X}_n$ respectively such that $B \cap Y = \emptyset$, $B \cap Z = \emptyset$, and $|B| + |Y| + |Z| = 4m$.
- When $|A| + \left|\hat{X}_1\right| + \left|\hat{X}_5\right| > 4m$, let B, Y, Z be arbitrary subsets of $A, \hat{X}_1, \hat{X}_5$ respectively, such that $|B| + |Y| + |Z| = 4m$.

In either case, we may apply Lemma 4.5.3 to obtain a list assignment L' on the shorter path P' such that:

- $|L'(v)| = 4m$ for all $v \in V(P')$, and
- $(A', \hat{X}'_1, \hat{X}'_3) = (B, Y, Z)$, and
- $S_{L'}(P') = 8m$.

We extend L' to all of G' by defining $L'(v) = L(v)$ for $v \notin V(P')$.

Let $G_0 = G' - V(P') = G - V(P)$, and let w and z be the neighbors of v'_1 and v'_3 in G_0, respectively. Since G' is $(4m:2m)$-choosable, Lemma 4.2.7 says there is a proper $(L':2m)$-coloring ϕ of G_0 such that $\mathrm{dam}_{L',P'}(\phi(w), \phi(z)) \leq 2m$. By the construction of (B, Y, Z) together with Lemma 4.2.10 and Lemma 4.2.11, this implies

$$\mathrm{dam}_{L,P}(\phi(w), \phi(z)) \leq S_L(P) - 10m.$$

Applying Lemma 4.2.7 in the other direction, we see that G is $(L:2m)$-colorable. Since L was arbitrary, G is $(4m:2m)$-choosable. □

Proof of Theorem 4.5.1. There are only finitely many bipartite 3-choosable-critical graphs that are minimal with respect to the reduction of Lemma 4.5.4. In particular, all such graphs have at most 14 vertices, the

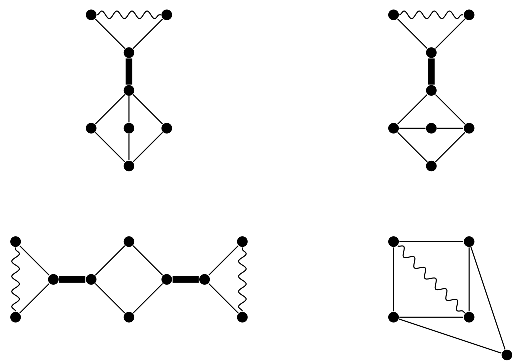


Figure 4.4: Exceptional graphs in Conjecture 4.6.1. Wavy lines represent paths with any odd number of vertices. Thick lines represent paths of with any nonnegative number of edges.

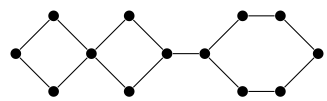


Figure 4.5: One possible realization of the lower-left graph in Figure 4.4.

largest being $\Theta_{5,5,5}$. Applying Theorem 4.5.2, we obtain a number $f(14)$ such that if $k/4$ is divisible by all numbers up to $f(14)$, then all minimal bipartite 3-choosable-critical graphs are $(4k : 2k)$ choosable. In particular, fixing the smallest such k and applying Lemma 4.5.4, we see that all bipartite 3-choosable-critical graphs are $(4mk : 2mk)$-choosable for all m. $\square$

4.6 Characterizing the $(4 : 2)$-Choosable Graphs: A Conjecture

Having determined which 3-choosable-critical graphs are $(4 : 2)$-choosable, the next natural step in investigating $(4 : 2)$-choosability is to characterize *all* $(4 : 2)$-choosable graphs, mirroring Rubin's characterization of the 2-choosable graphs [15]. As Theorem 4.1.2 shows, the $(4 : 2)$-choosable graphs have considerably more variety than the 2-choosable graphs, so the proof of any such characterization is likely to be much more involved than Rubin's proof.

Rubin observed that G is 2-choosable if and only if its core is 2-choosable, and the same observation holds for $(4 : 2)$-choosability. It clearly also suffices to consider only connected graphs, so we restrict to the case where G is connected with minimum degree at least 2.

Conjecture 4.6.1. *If G is a connected graph with $\delta(G) \geq 2$, then G is $(4 : 2)$-choosable if and only if one of the following holds:*

- *G is 2-choosable, or*

- *G is one of the* 3*-choosable-critical graphs listed in Theorem 4.1.2, or*

- *G is one in one of the families of graphs shown in Figure 4.4.*

Figure 4.4 indicates several multi-parameter families of graphs; Figure 4.5 shows an example of how to interpret this notation.

Conjecture 4.6.1 is supported by substantial evidence. Through computer search, we determined that among all graphs with at most nine vertices, only the graphs given by Conjecture 4.6.1 are $(4:2)$-choosable. It appears that all graphs with more vertices are either one of the $(4:2)$-choosable graphs listed in Conjecture 4.6.1, or contain some subgraph already known to be non-$(4:2)$-choosable.

A list of "small" minimal non-$(4:2)$-choosable graphs, each with a nonchoosable list assignment, is given in Figure 4.6. Each of the list assignments was found by computer search. The variety of these graphs represents a significant obstruction to any proof of Conjecture 4.6.1, which would seem to require a correspondingly complex structure theorem: having identified some family $\mathcal{F}$ which one believes to be the family of $(4:2)$-choosable graphs, one must show that every graph not in $\mathcal{F}$ contains some non-$(4:2)$-choosable graph, and with so many minimal non-$(4:2)$-choosable graphs, such a proof seems likely to be complex. While we believe that such a proof could be found, it would likely be quite long and beyond the scope of this chapter.

The computer analysis to show that the graphs listed Conjecture 4.6.1 are $(4:2)$-choosable is based on Lemma 4.2.7. Each of the graphs in Figure 4.4 has a small set of vertices X such that $G - X$ is a linear forest. Rather than generating all list assignments for the entire graph G, it suffices to generate all list assignments for X, and for each list assignment, to generate the possible tuples $(A, \hat{X}_1, \hat{X}_n)$ for each of the paths in $G - X$. For each such tuple, we then search for a partial coloring ϕ of $G[X]$ that satisfies the hypothesis of Lemma 4.2.7. For the graphs shown in Figure 4.4, our computer program found such a partial coloring for every possible choice of tuple. It is easy to manually check that each partial coloring indeed satisfies the hypothesis of the lemma, but we must trust that the computer program has successfully generated all possible combinations of tuples. The combinations of tuples are generated using McKay and Piperno's program `genbg`, part of the `nauty` suite [29], so the correctness of our program depends on the correctness of `nauty` as well as the correctness of our own implementation.

However, we have not been able to find a human-readable proof that the exceptional graphs in Conjecture 4.6.1 are indeed $(4:2)$-choosable, nor have we been able to prove the structure theorem alluded to above. Without a formal proof that the exceptional graphs are $(4:2)$-choosable, we retain a small amount of skepticism regarding that claim.

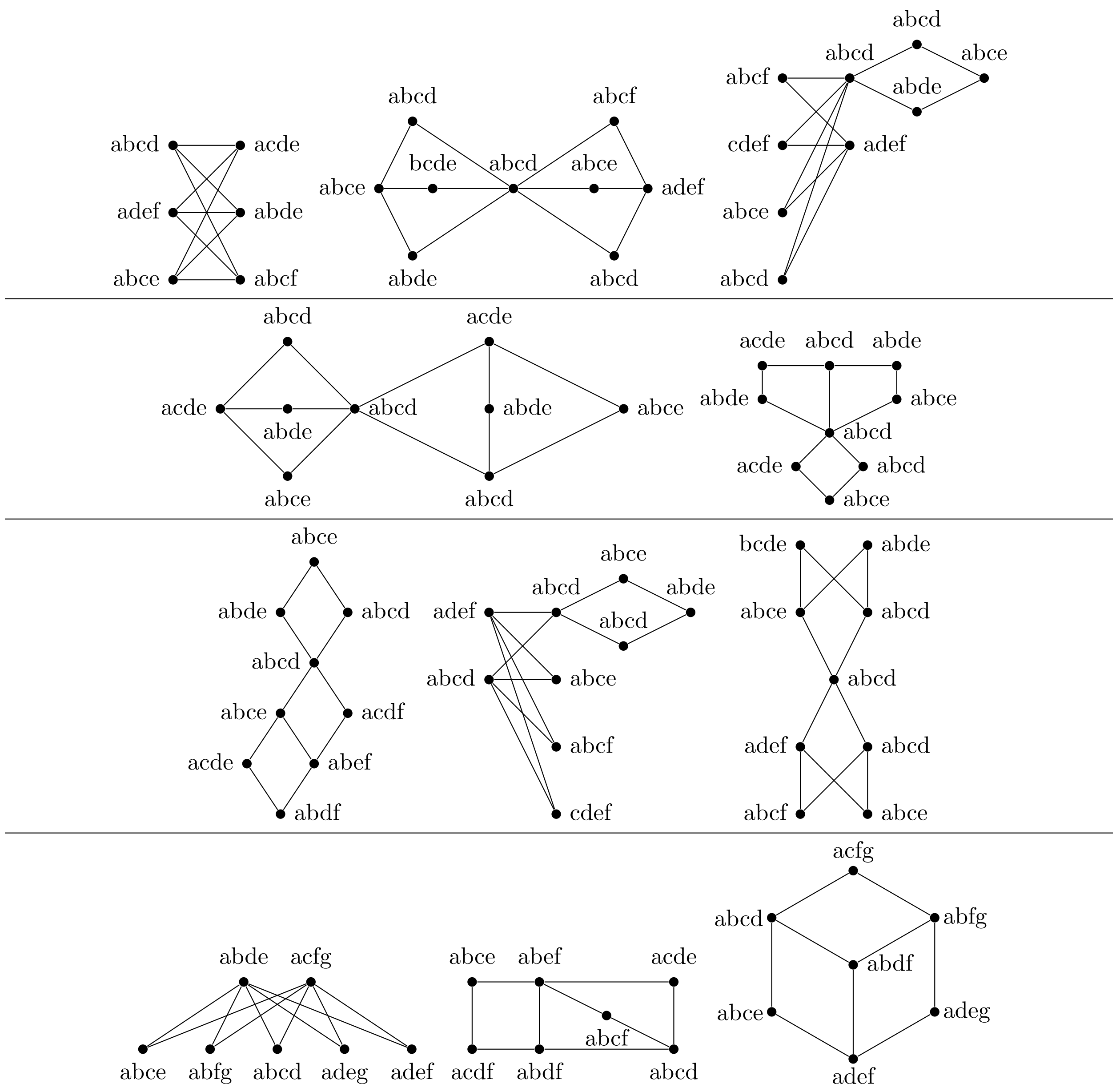


Figure 4.6: Some non-$(4:2)$-choosable graphs.

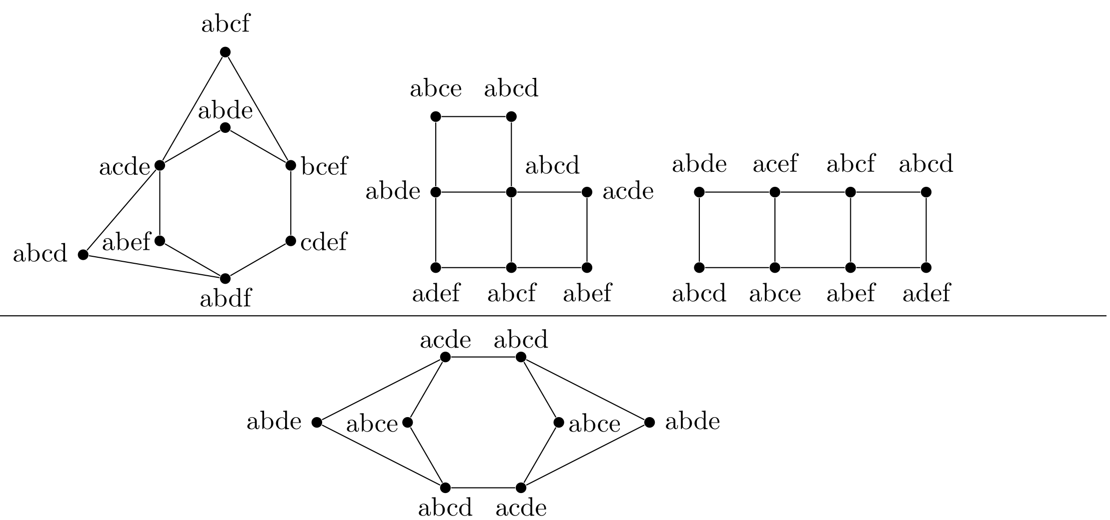


Figure 4.6: Some non-$(4:2)$-choosable graphs.

Chapter 5

Revolutionaries and Spies

This chapter is based on joint work with Jane V. Butterfield, Daniel W. Cranston, Douglas B. West, and Reza Zamani that appears in [4].

5.1 Introduction

Revolutionaries and Spies is a game of perfect information played on a graph by two opposing teams, a team of r revolutionaries and a team of s spies. At all times during the game, each revolutionary and each spy occupies a vertex of the graph. A vertex with m revolutionaries is a *meeting*; a meeting that also contains no spy is an *unguarded meeting.*

At the beginning of the game, each revolutionary chooses a vertex of the graph to occupy; multiple revolutionaries may occupy the same vertex. The spies then do the same. Play then proceeds as follows: if there is an unguarded meeting, then the revolutionaries win and the game is over. Otherwise, the game proceeds in *rounds*, a round consisting of the following steps:

- Each revolutionary has a chance to move, and may stay at its current for or move to a vertex adjacent to its current position.
- Each spy has a chance to move, in the same manner.
- At the end of the round, if there is still an unguarded meeting, then the game ends and the revolutionaries win. Otherwise, the game proceeds into a new round.

The revolutionaries win if there is ever an unguarded meeting at the end of a round, while the spies win by prolonging the game indefinitely.

Some readers may be uncomfortable with the fact that the spies never "win" in any finite period of time. From a purely formal perspective, this is not a problem: we can consider a play of any game as a potentially-infinite sequence of rounds, and the rules of the game merely need to assign a winner to each sequence. In the case of Revolutionaries and Spies, the sequences for which the spies win are precisely the infinite sequences. This is the most general solution to the definitional problem, since it keeps the game well-defined even on infinite graphs.

However, when the game is played on finite graphs (as with all graphs we consider here), we can express the rules of the game without recourse to infinite sequences. We modify the rules as follows: at the beginning of the game, the revolutionaries name a positive integer k and declare that they can win the game within k moves. We then say that the spies win if they can last k moves without allowing the revolutionaries to win. Since there are only finitely many board states on an n-vertex graph (certainly at most $(rs)^n$ states), this formulation is equivalent to the original formulation of the rules.

Definition 5.1.1. When G is a graph and m, r, s are nonnegative integers, $\mathrm{RS}(G, m, r, s)$ denotes the game of Revolutionaries and Spies played on the graph G with r revolutionaries, s spies, and meeting size m.

Definition 5.1.2. When G is a graph and m, r are nonnegative integers, let $\sigma(G, m, r)$ denote the smallest integer s such that the spies win $\mathrm{RS}(G, m, r, s)$.

Revolutionaries and Spies was originally defined in the mid-1990s by Beck (unpublished). Howard and Smyth [23] studied the game on integer lattices. Cranston, Smyth, and West [7] determined $\sigma(G, m, r)$ in the case where G is a tree or a unicyclic graph. Mitsche and Prałat [30] proved bounds on $\sigma(G, m, r)$ for random graphs which are asymptotically matching (up to a constant factor) for a large range of edge probabilities.

Butterfield, Cranston, Puleo, West, and Zamani [4] studied the game on a variety of graphs. In this chapter, we will discuss and expand on some of the results of that paper.

The rest of the chapter is structured as follows. In Section 5.2 we state some general bounds that hold for all graphs. These bounds help place the remaining results into a sensible context, so that we have a sense of what it means for a graph to be "good for the revolutionaries" or "good for the spies". Section 5.2 also introduces some notation that we will use in the rest of the chapter.

A *dominating set* in a graph is a set of vertices X such that every vertex either lies in X or has a neighbor in X. A result from [4] states that the spies can use a dominating set of size γ to win with at most $\gamma \lfloor r/m \rfloor$. In Section 5.3, we construct an infinite family graphs for which this bound is sharp.

In Section 5.4, we study the d-dimensional hypercube Q_d. We use a probabilistic argument to show that these graphs are "good for revolutionaries".

A *split graph* is a graph whose vertex set can be partitioned into sets Q and S so that Q is a clique and S is an independent set. In Section 5.8, we discuss split graphs, and give a spy strategy for these graphs.

5.2 General Bounds and Notation

In this section we give some general bounds for Revolutionaries and Spies that hold in all graphs, and we will also introduce some notation used in the rest of the chapter. These bounds arise from very simple strategies and provide context for bounds in specific graph families.

The following result first appeared (in its present form) in Cranston–Smyth–West [7], although an earlier version existed in Howard–Smyth [23].

Proposition 5.2.1. *On any graph G and for any positive m and r,*

$$\min\{|V(G)|, \lfloor r/m \rfloor\} \leq \sigma(G, m, r) \leq \min\{|V(G)|, r - m + 1\}.$$

Proof. If $s < |V(G)|$ and $s < \lfloor r/m \rfloor$, then the revolutionaries can start at $s + 1$ distinct vertices, placing m revolutionaries on each vertex, and placing the remaining revolutionaries arbitrarily. This creates $s + 1$ meetings, of which the spies can only guard s; hence, the revolutionaries win after the setup phase.

If $s \geq |V(G)|$, the spies can simply place one spy on each vertex of G and never move. Hence the revolutionaries can never form an unguarded meeting. If $s \geq r - m + 1$, then the spies can choose $r - m + 1$ distinct revolutionaries and assign a spy to follow each one. This guarantees that throughout the game, any vertex with m revolutionaries also has a spy. $\square$

Theorem 5.2.1 suggests that graphs for which $\sigma(G, m, r)$ is close to $r - m$ are "good for the revolutionaries", needing close to the maximum possible number of spies, while graphs for which $\sigma(G, m, r)$ is close to r/m are "good for the spies", needing close to the minimum possible number of spies. In particular, these bounds suggest that we should ask how $\sigma(G, m, r)$ varies in m when r is fixed and "reasonably small" relative to $|V(G)|$.

It is not difficult to find graphs that are as good as possible for the spies: if G is a clique or, more generally, if G has a dominating vertex, then $\sigma(G, m, r) = \min\{|V(G)|, \lfloor r/m \rfloor\}$. Finding graphs that are good for revolutionaries is somewhat harder, but we have the following construction, which in fact yields a split graph:

Theorem 5.2.2. *If $m_0 \geq 1$ and $r_0 \geq 1$, then there exists a split graph G such that $\sigma(G, m, r) = r - m + 1$ whenever $m \leq r \leq r_0$ and $m \leq m_0$.*

Proof. Let X be a set of size r_0, and let $Y = \binom{X}{1} \cup \binom{X}{2} \cup \cdots \cup \binom{X}{m_0}$. Construct a graph G with $V(G) = X \cup Y$ by making X a clique, making Y an independent set, and making vertices $x \in X$ and $y \in Y$ adjacent whenever y contains x (considering y as a subset of X).

Suppose $m \leq r \leq r_0$; we show that r revolutionaries beat $r - m$ spies. The revolutionaries start at distinct vertices of X. Given any spy response, let q be the number of spies placed in Y. Since there are only $r - m$ spies in total, there are at least $m + q$ revolutionaries with no spy on their vertex; call these the *unguarded revolutionaries*. With $m + q$ unguarded revolutionaries in total, there is some set of m unguarded revolutionaries such that the corresponding m-set in Y has no spy on it. These revolutionaries form a meeting on the next turn, and there is no adjacent spy to guard it. $\square$

We will revisit split graphs in Section 5.8.

Now we introduce notation. We will often need to refer to the number of revolutionaries or the number of spies that are present in a vertex set at some point in time. When $X \subseteq V(G)$, we write r_X to denote the number of revolutionaries in X at a given time, and we write s_X to denote the number of spies in X. When X consists of a single vertex v, we suppress the braces and write r_v or s_v.

We refine this notation by addressing another frequent problem. Often, we will need to consider the number of revolutionaries at the *beginning* or *end* of some particular round. In this case, we typically write r_X for the number of revolutionaries in X at the beginning of the round (before anyone has moved) and write r'_X for the number of revolutionaries in X at the end of the round. The spy analogues s_X and s'_X are defined similarly.

5.3 Domination Sharpness

A *dominating set* in a graph is a set of vertices X such that every vertex either lies in X or has a neighbor in X. The *domination number* of a graph is the size of a smallest dominating set.

The following theorem was proved in [4], and also appears in the thesis of coauthor Reza Zamani:

Theorem 5.3.1. *When G is a graph with domination number gamma, the spies win* $\mathrm{RS}(G, m, r, \lfloor r/m \rfloor)$.

The basic idea of the proof is to reduce to the case $\gamma = 1$ and show that $\gamma \lfloor r/m \rfloor$ spies suffices when G has a dominating vertex, using a Hall's Theorem argument. In the general case, we assign $\lfloor r/m \rfloor$ spies to each vertex v in the dominating set, and these spies play the $\gamma = 1$ strategy for the subgraph $G[N[v]]$, with the spies assigned to different vertices playing independently.

In this section, we will prove that this bound is sharp in the following sense:

Theorem 5.3.2. *For any positive integers γ and t_0, there exists a graph G with domination number at most γ such that for all $t \in \{2, \ldots, t_0\}$, the revolutionaries win* $\mathrm{RS}(G, m, r, \gamma \lfloor r/m \rfloor - 1)$*, where $m = \gamma$ and $r = \gamma t - 1$.*

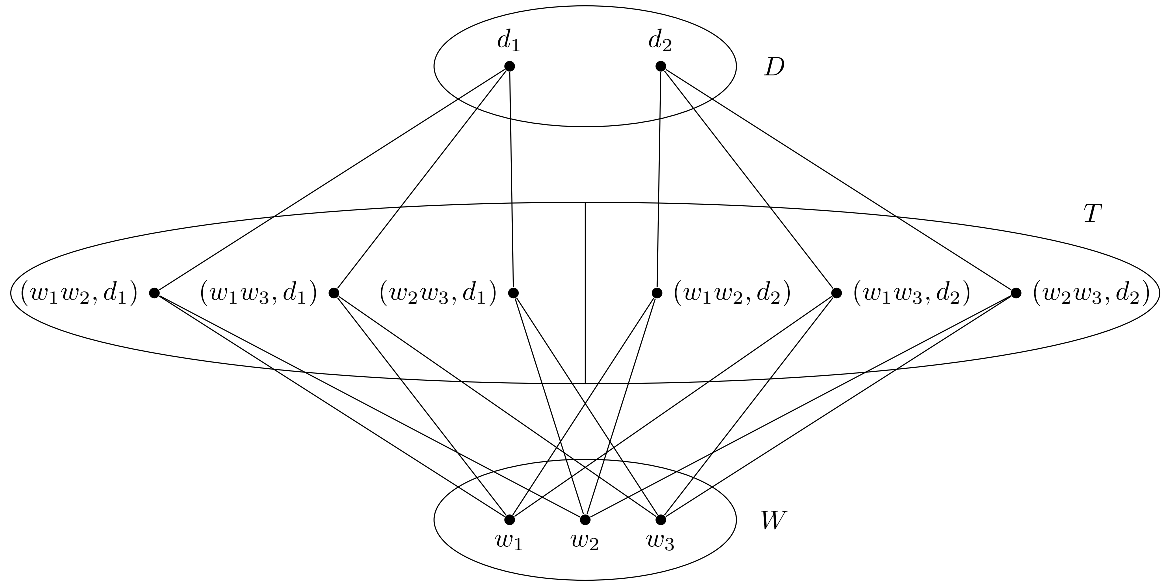


Figure 5.1: Example of the construction in Theorem 5.3.2, with $\gamma = t_0 = 2$. To reduce clutter, edges joining D and W are omitted.

Proof. Our construction is based on the simpler construction in Theorem 5.2.2. Let W and D be disjoint sets with $|W| = \gamma t_0 - 1$ and $|D| = \gamma$. Let $T = \binom{W}{\gamma} \times D$. Define a graph G with vertex set $W \cup D \cup T$ and edge set defined as follows:

- For all $w \in W$ and $d \in D$, put $w \leftrightarrow d$;
- For $w \in W$ and $(X, d) \in T$, put $w \leftrightarrow (X, d)$ if and only if $w \in X$;
- For $d \in D$ and $(X, d') \in T$, put $d \leftrightarrow (X, d')$ if and only if $d = d'$.

The construction is illustrated in Figure 5.1.

It is clear that D is a dominating set in G of size γ. Let any $t \in \{2, \ldots, t_0\}$ be given. We claim that $\gamma t - 1$ revolutionaries win against $\gamma(t-1) - 1$ spies when the meeting size is γ. In fact, the revolutionaries will win in Round 1 by making a meeting in T. The revolutionaries start by placing one revolutionary on each of $\gamma t - 1$ distinct vertices of W. Consider any response by $\gamma(t-1) - 1$ spies.

Say that a spy *watches* the T-vertices in its closed neighborhood (the T-vertices are the only vertices we care about for the sake of this definition). If the revolutionaries form a meeting at v, then the only way for the spies to respond without losing is to move some spy that watches v onto the meeting. A spy on a D-vertex v watches all T-vertices of the form (X, v), and a spy on a T-vertex watches only that T-vertex.

For each $v \in D$, let $f(v) = \{v\} \cup N_T(v)$. The sets $f(v)$ partition $D \cup T$, since every T-vertex is adjacent to exactly one D-vertex. Let v^* be the vertex of d minimizing $s_{f(v^*)}$, so that $s_{D \cup T} \geq \gamma s_{f(v^*)}$. Since there are $\gamma(t-1)-1$ spies in total, we have

$$s_W + \gamma s_{f(v^*)} \leq \gamma(t-1)-1.$$

Rearranging, we have

$$s_W \leq \gamma(t - s_{f(v^*)} - 1) - 1.$$

Since all revolutionaries in W are on distinct vertices, there are at least $r - s_W$ revolutionaries in W with no spy on their vertex. Call these the *unguarded revolutionaries*. From the previous inequality, we have

$$r - s_W \geq \big(\gamma t - 1\big) - \big(\gamma(t - s_{f(v^*)} - 1) - 1\big) = \gamma(s_{f(v^*)} + 1),$$

and so we can split the unguarded revolutionaries into $s_{f(v^*)} + 1$ disjoint sets of size γ. Write these sets $X_1, \ldots, X_{s_{f(v^*)}+1}$. Moving the revolutionaries in X_i to the T-vertex (X_i, v^*), we form $s_{f(v*)} + 1$ distinct meetings, each of which is only watched by the spies in $f(v^*)$. Since there are only $s_{f(v^*)}$ of these spies, the spies cannot move to guard all of the newly-formed meetings, and they will lose after their move. □

We note briefly here that although the graph as defined makes D and W independent sets, our proof never made use of this fact: the adjacencies inside D and the adjacencies inside W are irrelevant. Thus, the graph could be modified to make $D \cup W$ a clique, if desired, and the proof would still work. Since T is an independent set, this modified graph is a split graph.

5.4 Hypercubes are Good for Revolutionaries

The *d-dimensional hypercube graph*, written Q_d, has several different definitions. For our purpose, the following definition will be most convenient: given a set X, let $\mathcal{P}(X)$ be the collection of all subsets of X. We define Q_d as the graph with vertex set $\mathcal{P}([d])$ where two sets v, w are adjacent if they differ by a single element (that is, if $|v \oplus w| = 1$). This definition allows us to apply set operations like union and intersection to the vertices of Q_d. The *weight* of a vertex is its size as a subset of $[d]$; thus, we write $|v|$ to denote the weight of v.

As the following lemma shows, the distance between two vertices in Q_d is just the size of their symmetric difference as sets:

Lemma 5.4.1. *The distance $d(v, w)$ between any vertices $v, w \in V(Q_d)$ is given by*

$$d(v, w) = |v \oplus w| \,.$$

Proof. Writing the symmetric difference $v \oplus w$ as $\{x_1, \ldots, x_t\}$ and defining

$$\begin{aligned} v_0 &= v, \\ v_i &= v_{i-1} \oplus \{x_i\} \qquad (i \in [t]), \end{aligned}$$

we see that $v_0 v_1 \cdots v_t$ is a v, w-path in Q_d with t edges. Hence $d(v, w) \leq |v \oplus w|$. On the other hand, if $v_0 v_1 \cdots v_t$ is a shortest v, w-path in Q_d, then the triangle inequality yields

$$\begin{aligned} |v \oplus w| = |v_0 \oplus v_t| &= |v_0 \oplus (v_1 \oplus v_1) \oplus \cdots \oplus (v_{t-1} \oplus v_{t-1}) \oplus v_t| \\ &= |(v_0 \oplus v_1) \oplus (v_1 \oplus v_2) \oplus \cdots \oplus (v_{t-1} \oplus v_t)| \\ &\leq |v_0 \oplus v_1| + \cdots + |v_{t-1} \oplus v_t| \\ &\leq t. \end{aligned}$$

Hence $d(v, w) \geq |v \oplus w|$. □

When $m = 2$, the graph Q_d is as good as possible for revolutionaries:

Theorem 5.4.2. *If $d \geq r$, then $\sigma(Q_d, 2, r) = r - 1$.*

Proof. It suffices to show that $r - 2$ spies cannot win. The revolutionaries start on the weight-1 vertices $\{1\}, \ldots, \{r\}$, threatening meetings at $\emptyset$ and the vertices of weight 2. Let t be the number of revolutionaries not sharing their vertex with a spy. Say that a spy *watches* the vertices in its closed neighborhood; if there is some unwatched vertex with two revolutionaries in its closed neighborhood, then the revolutionaries can make a meeting there and win.

The revolutionaries threaten $\binom{t}{2}$ meetings on the vertices of weight 2 that are not watched by spies on vertices weight 1. Assume that the spies have a winning response; in particular, this response watches all threats on the weight-2 vertices.

A spy on a vertex of weight 2 watches at most one of these meetings, a spy on a vertex of weight 3 watches at most three of these meetings, and a spy on a vertex of any other weight watches none of these meetings. For the spies to win, they must watch all of these meetings. Say that a spy is a *rogue spy* if it is not on a vertex of weight 1. There are at most $s - (r - t)$ rogue spies; since $s = r - 2$, there are at most

$t-2$ rogue spies. Since all threats are watched, this implies $3(t-2) \geq \binom{t}{2}$. When $s = r-2$, this is only possible if $3 \leq t \leq 4$; furthermore, to watch all threats, there are exactly $t-2$ rogue spies. We split into cases according to t.

Case 1: $t = 4$. In this case, there are two rogue spies. We may reindex so that the singletons $\{i\}$ for $i \in [4]$ have no spy. The revolutionaries threaten meetings at the six weight-2 vertices in $\binom{[4]}{2}$. A rogue spy watches at most 3 meetings, and for such a spy to watch exactly three meetings, its vertex must be a weight-3 subset of [4]. Since the spies watch all six threats, the two rogue spies must be on weight-3 subsets of [4]. Any two such subsets intersect in at least two elements, so if v and w are the vertices occupied by the rogue spies, then $|v \cap w| \geq 2$, so that v and w have some common neighbor of weight 2. Watching some threat twice implies that some other threat is not watched at all, since each spy watches at most three threats. This contradicts the assumption that all threats are watched.

Case 2: $t = 3$. In this case, there is one rogue spy, and it watches exactly three meetings on the vertices of weight 2. Without loss of generality, we may reindex so that the rogue spy is on the vertex $\{1, 2, 3\}$ and the remaining spies are on the singletons $\{i\}, \ldots, \{r\}$.

In Round 1, the revolutionaries respond by moving the revolutionaries on $\{1\}$ and $\{2\}$ to the vertex $\emptyset$, leaving all other revolutionaries where they are. The spies must respond by guarding the meeting at $\emptyset$, and can only do so with a spy on a singleton, say the spy at $\{j\}$. At the beginning of Round 2, every spy other than the spy at $\{j\}$ has distance at least 3 from the vertex $\{3, j\}$, so at the end of Round 1, no spy is adjacent to $\{3, j\}$. During Round 2, the revolutionaries at $\{3\}$ and $\{j\}$ then move to form a meeting at $\{3, j\}$, and the spies cannot guard this meeting. $\square$

In this section, we prove a version of this result that holds for $m > 2$:

Theorem 5.4.3. *If $d \geq r$, then $\sigma(Q_d, m, r) \geq r - 38.73m$.*

We rephrase the problem as a problem in extremal set theory, which we approach using the probabilistic method. We will need the following lemma:

Lemma 5.4.4. *Let $v \in Q_d$. A vertex w of weight m satisfies $d(v, w) \leq m-1$ if and only if it satisfies $|v \cap w| \geq \frac{|v|+1}{2}$.*

Proof. We have $d(v, w) \leq m$ if and only if $|w| + |v| - 2|w \cap v| \leq m - 1$. When $|w| = m$, this condition rearranges to $|w \cap v| \geq \frac{|v|+1}{2}$. $\square$

Our main tool is the following lemma:

Lemma 5.4.5. *Let S be a set of vertices in Q_n each having weight at least 2. If $|S| \leq n$ and $n \geq 38.73m$, then there exists a point $w \in \binom{[n]}{m}$ such that $d(v, w) \geq m$ for all $v \in S$.*

Proof. Let p be a nonzero probability that we will specify later. Construct a random subset $I \subseteq [n]$ by independently placing each element of $[n]$ into I with probability p. Say that a point $v \in S$ is *avoided* if $|v \cap I| < \frac{|v|+1}{2}$, and let A_v be the event that v is avoided. Note that A_v occurs if and only if I falls into the downset $\{J \subseteq [n] \colon\ |v \cap J| \leq \frac{|v|+1}{2}\}$. Our goal is to show that with an appropriately chosen p, there is positive probability that all points in S are avoided and that $|I| \geq m$, so that taking any m-set in I yields a vertex with the desired properties.

Fix a vertex $v \in S$, and let $k = \lceil(|v| + 1)/2\rceil$, so that $k \geq 2$ and $|v| \leq 2k - 1$. Avoiding v means that fewer than k elements of v show up in I, so we have

$$\mathbb{P}[A_v] \geq \mathbb{P}[\mathrm{Bin}(2k - 1, p) < k],$$

with equality holding only when $|v|$ is odd. We will first argue that for $p < 1/2$, the probability $\mathbb{P}[\mathrm{Bin}(2k - 1, p) < k]$ is maximized when $k = 2$.

Consider a series of $2k + 1$ trials, each with success probability p. Let the random variable X denote the number of successes among the first $2k - 1$ trials, and let Y denote the number of successes among all $2k + 1$ trials, so that $X \sim \mathrm{Bin}(2k - 1, p)$ and $Y \sim \mathrm{Bin}(2k + 1, p)$. Let B_X denote the event that $X \leq k - 1$, and let B denote the event that $Y \leq k$. Our goal is to show that $\mathbb{P}[B_X] \leq \mathbb{P}[B_Y]$.

It suffices to compare $\mathbb{P}[B_X \wedge \overline{B}]$ and $\mathbb{P}[\overline{B_X} \wedge B_Y]$. For the event $B_X \wedge \overline{B_Y}$ to occur, we must have exactly $k - 1$ successes in the first $2k - 1$ trials, followed by two successes. For the event $\overline{B_X} \wedge B_Y$ to occur, we must have exactly k successes in the first $2k - 1$ trials, followed by two failures. Thus,

$$\begin{aligned}
\mathbb{P}[B_X \wedge \overline{B_Y}] &= \binom{2k-1}{k-1} p^{k-1}(1-p)^k \cdot p^2, \text{ and} \\
\mathbb{P}[\overline{B_X} \wedge B_Y] &= \binom{2k-1}{k} p^k (1-p)^{k-1} \cdot (1-p)^2.
\end{aligned}$$

Since $p < 1/2$, it follows that

$$\frac{\mathbb{P}[B_X \wedge \overline{B_Y}]}{\mathbb{P}[\overline{B_X} \wedge B_Y]} = \frac{p}{1-p} < 1,$$

so $\mathbb{P}[B_X \wedge \overline{B_Y}] < \mathbb{P}[\overline{B_X} \wedge B_Y]$. This yields $\mathbb{P}[B_X] < \mathbb{P}[B_Y]$, as desired. Thus, $\mathbb{P}[\mathrm{Bin}(2k - 1, p) < k]$ is maximized when $k = 2$. It follows that for any vertex $v \in S$,

$$\mathbb{P}[A_v] \geq \mathbb{P}[\mathrm{Bin}(3, p) < 2] = (1 - p)^2(1 + 2p).$$

Let $q = (1-p)^2(1+2p)$.

Since each event A_v corresponds to I falling into a downset, the FKG Inequality (in particular, its corollary Lemma 1.5.3) gives us

$$\mathbb{P}\left[\bigwedge_{v\in S} A_v\right] \geq q^n = e^{n\ln q}.$$

On the other hand, let $X = |I|$. Let α be a positive constant less than 1, to be determined later. Chernoff's Inequality tells us that when $m \leq \alpha pn$,

$$\begin{aligned}\mathbb{P}[X < m] &= \mathbb{P}[X - \mathbb{E}[X] < m - np] \\ &\leq e^{-(m-np)^2/(2pn)} \\ &\leq e^{-(1-\alpha)^2 np/2}.\end{aligned}$$

Our goal is to choose p and α so that $\mathbb{P}[X < m] < \mathbb{P}\left[\bigwedge_{v\in S} A_v\right]$, that is, so that $n\ln(q) > -(1-\alpha)^2 np/2$. Canceling n from both sides, it is sufficient that $\ln(q) > -(1-\alpha)^2 p/2$, where $q = (1-p)^2(1+2p)$, as before. We wish to maximize αp subject to this constraint, since we want the inequality $m \leq \alpha pn$ to hold for small values of n.

Setting the left and right sides of this inequality to be equal to each other yields a transcendental equation, so there is little hope of an exact solution. We therefore seek a numerical solution. Taking $\alpha = .324722$ and $p = .079532$ satisfies the inequality and yields $\alpha p \approx .0258259$. Since $1/.0258259 \leq 38.73$, the inequality $m \leq \alpha pn$ holds when $n \geq 38.73m$. For such n, there is positive probability that all points in S are avoided and that $|I| \geq m$, so there is a vertex w with the desired properties. □

Corollary 5.4.6. *If $d \geq r$, then $\sigma(Q_d, m, r) \geq r - 38.73m$.*

Proof. Assuming that $s \leq r - 38.73m$, we show that the revolutionaries win. The revolutionaries start at the singletons $\{1\}, \ldots, \{r\}$. Given any spy response, let n be the number of revolutionaries with no spy on their vertex. Relabel the hypercube so that the singletons $\{1\}, \ldots, \{n\}$ are not occupied by spies, and let S be the set of spies on vertices of weight at least 2. We have

$$n \geq s - r \geq 38.73m,$$

but also $|S| \leq n$, since each vertex in $[r] - [n]$ has at least one spy. (Indeed $|S| \leq n - 38.73m$.) This satisfies the hypotheses of Lemma 5.4.5 and hence there is some vertex w of weight m, contained in $[n]$, such that $d(v, w) \geq m$ for all $v \in S$. If v is a vertex outside S, then v is disjoint from w, so $d(v, w) = |v| + m \geq m$.

Thus, every spy is at least m rounds away from w.

Hence the revolutionaries can make a meeting at w in $m-1$ rounds, and the spies will not be able to guard this meeting. $\square$

Computer experiments suggest that if $d \geq r$ and $m \geq 3$, then $\sigma(Q_d, m, r) \geq r - 2m$. However, we have not been able to prove a sharper lower bound than that of Theorem 5.4.3.

5.5 Spy Strategies: General Principles

For the rest of this chapter, we will concentrate on finding strategies for the spies on various types of graph. Finding a strategy for the spies is typically more difficult than finding a strategy for the revolutionaries: while a strategy for the revolutionaries may be guaranteed to win the game in very few rounds, a strategy for the spies must be able to survive indefinitely. In this section, we will formulate some general principles that we apply to each graph class considered later.

Fix an order on the spies; we will call one spy *smaller* or *larger* than another with respect to this ordering. A spy is *free* if it is either not the smallest spy on its vertex (with respect to this ordering), or if there is no meeting on its vertex. A spy that is not free is *bound to* the meeting on its vertex; if multiple spies are on a vertex with a meeting, only the smallest of them is bound to the meeting. The number of free spies in some set X at at the beginning of a round is written $\hat{s}_X$, and the number at the end of the round is written $\hat{s}'_X$. We write $\hat{s}$ for $\hat{s}_{V(G)}$.

Fix an order on the revolutionaries. A revolutionary x is *bound* if there is a meeting on its vertex and x is among the m smallest revolutionaries on the vertex (thus, every vertex has either 0 or m bound revolutionaries). A revolutionary is *free* if it is not bound. A revolutionary is *free* if there is no meeting on its vertex, or if it is not among the smallest m revolutionaries on the vertex. The notation $\hat{r}_X$, $\hat{r}'_X$, and $\hat{r}$ is defined as before.

In a position where every meeting is covered, every meeting results in m bound revolutionaries and 1 bound spy. Hence, when every meeting is covered, $r - m\hat{r} = s - \hat{s}$.

Each spy strategy will rely on a notion of a *stable position*. The definition of a stable position will be slightly different for different graph classes, but all of the definitions have essentially two parts:

1. All meetings are covered, and

2. After any move by the revolutionaries, the spies can move to re-establish a stable position.

Choosing a suitable definition of "stable" is like trying to strengthen an induction hypothesis: a strict definition of stability grants a powerful hypothesis for proving both parts above, but also demands a stronger conclusion in the second part.

The following lemma will be useful:

Lemma 5.5.1. *If $\hat{s}_{N[v]} \geq \hat{r}_{N[v]}/m$ for all $v \in V(G)$, then after any move by the revolutionaries, the spies can move to cover all meetings. Furthermore, the spies can move to cover all meetings in such a way that no spy who moved is free afterwards.*

Proof. First we argue that the spies have *some* move that covers all meetings, without respect to the added condition about free spies. We then show that choosing the move "optimally" guarantees the added condition as well.

We use a Hall's Theorem argument, applied to an auxiliary bipartite graph whose partite sets represent spies and meetings. Let X be the set of all meetings, and let Y be the set of spies. Construct an X,Y-bigraph H by making $x \in X$ and $y \in Y$ adjacent when the spy y can reach the vertex x in one move. We show that there is a matching that covers X. If we find such a matching then we are happy: sending each matched spy to the corresponding vertex covers all meetings. To apply Hall's Theorem, consider $X_0 \subseteq X$; we show that $|N_H(X_0)| \geq |X_0|$.

Let $Z = N_G[X_0]$. If Z contains b meetings at the start of the round, then

$$|X_0| \leq \frac{\hat{r}_Z + mb}{m} = \hat{r}_Z/m + b,$$

since revolutionaries starting outside $N_G[X_0]$ cannot reach X_0 in one move. We show that $|N_H[X_0]| \geq \hat{r}_Z/m + b$.

Every free spy at a vertex of $N_G[X_0]$, as well as every spy that started the round bound to a meeting in $N_G[X_0]$, can reach X_0 in one move. Since $\hat{s}_v \geq \hat{r}_{N_G[v]}/m$ for all $v \in X_0$, we have

$$\hat{s}_{X_0} = \sum_{v \in X_0} \hat{s}_v \geq \sum_{v \in X_0} \hat{r}_{N[v]}/m \geq \hat{r}_Z/m.$$

Since there were b meetings in $N_G[X_0]$ at the start of the round and all meetings are covered, there are exactly b spies that start the round bound to a meeting in $N_G[X_0]$. Hence

$$|N_H[X_0]| \geq \hat{r}_Z/m + b,$$

as desired. Therefore, there is some move that covers all the meetings.

Now we guarantee the added condition. Among all possible moves that cover all meetings, choose one that moves the fewest spies. If a spy is moved, it must be become bound at the end of the round: otherwise, all meetings are still covered when the spy does not move, contradicting the choice of the move. □

5.6 Spies with a Spanning Complete k-partite Graph

In this section, we will prove the following theorem:

Theorem 5.6.1. *If G has a spanning complete k-partite subgraph, then* $\sigma(G, m, r) \leq \left\lceil \frac{k}{k-1} \frac{r}{m} \right\rceil + k$.

Let $V_1, \ldots, V_k$ be the partite sets of the spanning complete k-partite subgraph, and let $r_i = r_{V_i}$ (and likewise for all other notation).

Say that a position is *stable* if:

- All meetings are covered, and
- $\hat{s} - \hat{s}_i \geq \hat{r}/m$ for each $i \in [k]$.

We need to show that, starting from a stable position, we can always re-establish stability at the end of a round:

Lemma 5.6.2. *If a position is stable at the beginning of the round, then after any revolutionary move, the spies can make a move that results in a stable position.*

Proof. Our move proceeds in two phases: in the first phase, we cover all the meetings. In the second phase, we restore stability.

Since the position is stable at the beginning of the round, we have $\hat{s}_i \geq \hat{r}/m$ for each $i \in [k]$. Thus, for $v \in V_i$, we have

$$\hat{s}_{N[v]} \geq \sum_{j \neq i} \hat{s}_j = \hat{s} - \hat{s}_i \geq \hat{r}/m \geq \hat{r}_{N[v]}/m.$$

By Lemma 5.5.1, there is some move that covers all meetings and does not move any spy that ends free. During the first phase, the spies make such a move.

During the second phases, move the free spies so that they are distributed as evenly as possible between the partite sets. We claim that the resulting position is stable. Since the free spies are distributed as evenly as possible, each partite set satisfies $\hat{s}_i \geq \lfloor \hat{s}/k \rfloor$. Thus, we need to show that $\lfloor \hat{s}/k \rfloor \geq \hat{r}/m$. Since

$r - m\hat{r} = s - \hat{s}$, we have

$$\begin{aligned}
\hat{s} &= s - (r - \hat{r})/m \\
&\geq \frac{k}{k-1}\frac{r}{m} + k - (r - \hat{r})/m \\
&\geq \frac{k}{k-1}\frac{r}{m} - \frac{k}{k-1}\frac{r-\hat{r}}{m} + k \\
&\geq \frac{k}{k-1}\frac{\hat{r}}{m} + k.
\end{aligned}$$

Thus, $\lfloor \hat{s}/k \rfloor \geq \hat{r}/(m(k-1))$, and so $\hat{s}_i \geq \hat{r}/(m(k-1))$ for each partite set i. This implies that $\hat{s} - \hat{s}_i = \sum_{j \neq i} \hat{s}_i \geq \hat{r}/m$, as desired. $\square$

To complete the proof of Theorem 5.6.1, we must show that the spies can initially establish a stable position after the revolutionaries' initial placement.

To handle this, the spies can imagine a fictitious first revolutionary move where all revolutionaries start out on a single vertex v. By placing a single spy on v and evenly distributing the remaining spies among the partite sets, the spies obtain a stable position:

- Every meeting is covered, since if $r \geq m$, then there is at least one spy, and that spy is on the same vertex as every revolutionary.
- Since $\hat{s} - \hat{s}_i \geq (s-1) - \left\lceil \frac{s-1}{k} \right\rceil \geq \frac{k-1}{k}(s-1) - 1 \geq \frac{r}{m}$, we have $\hat{s} - \hat{s}_i \geq \frac{\hat{r}}{m}$.

The imagined revolutionaries can then spend two turns moving to their real starting locations, and the spies can respond according to the lemma to maintain a stable position. Thus, the spies win.

5.7 Spies on a Random Graph

In this section, we consider the Erdős–Rényi random graph model, where an n-vertex graph $G(n,p)$ is randomly generated by making each possible edge present with some probability p. In general, the edge probability p is allowed to depend on n, but in this section, we only consider the case where p is a fixed constant in $(0,1)$. An event occurs *almost surely* if it occurs with probability tending to 1 as $n \to \infty$.

The name is a slight misnomer: this specific random graph model was first introduced by Gilbert [17], while Erdős and Rényi studied a very similar model where the number of edges m is fixed and a set of m edges is chosen at random [13, 14].

The game of Revolutionaries and Spies was also studied on random graphs by Mitsche and Prałat [30], who independently obtained stronger bounds than the ones presented here. Their proof uses more detailed properties of the structure of random graphs than the properties used here.

In this section, we prove that random graphs are spy-good in the following sense:

Theorem 5.7.1. *If G is a random graph with n vertices and edge probability p, then for any $\epsilon > 0$, almost surely the following holds: for all $m \geq 2$ and all $r \geq 0$,*

$$\sigma(G, m, r) \leq \max\left\{ \frac{1+\epsilon}{q}\frac{r}{m} \ ,\ \frac{r}{m} + \frac{\ln n}{2(1-1/(1+\epsilon))^2 q^2} \right\}.$$

We first identify a structural property of random graphs that holds almost surely – the property of being *q-common* – and then we prove that when s satisfies the bound above, the spies have a (randomized!) winning strategy on q-common graphs.

Definition 5.7.2. For $q \in (0,1)$, a graph is *q-common* if $\frac{|N(v)\cap N(w)|}{N(v)} \geq q$ for all distinct $v, w \in V(G)$.

Lemma 5.7.3. *For any $\epsilon > 0$, the random graph $G(n,p)$ is almost surely $(p-\epsilon)$-common.*

The proof of Lemma 5.7.3 is routine, technical, and uninteresting, so we defer it to the end of the section.

The proof that spies win on q-common graphs is similiar the proof in the previous section: we define a notion of stability, and, starting from a stable position, we apply Lemma 5.5.1 to cover all meetings, then move the resulting free spies to re-establish stability. In the rest of the section, we assume without explicit statement that G is q-common.

Say that a position is *stable* if:

- All meetings are covered, and
- $\hat{s}_{N[v]} \geq \hat{r}/m$ for all $v \in V(G)$.

Lemma 5.7.4. *If the position is stable at the beginning of the round, then after any revolutionary move, the spies can make a move that results in a stable position.*

Proof. As in the previous section, we divide the spy move into two phases: in Phase 1, we apply Lemma 5.5.1 to guard all meetings, without moving any spy that ends free. In Phase 2, we move the free spies to re-establish stability.

Since the position is stable at the beginning of the round, the hypotheses of Lemma 5.5.1 are satisfied. Thus, there is some move that covers all meetings and does not move any spy that ends free. During the first phase, the spies make such a move.

During the second phase, the free spies move to re-establish stability. We use a probabilistic argument to show that the spies have some move that re-establishes stability.

Consider the random move generated by having each free spy move to a randomly chosen neighbor, with all such choices made independently. For each $v \in V(G)$, let the random variable X_v be the number of spies that end in $N[v]$ after the random move. Since G is q-common, each individual spy has probability at least q of ending the round in $N[v]$. Furthermore, these events are independent, so X_v is a sum of $\hat{s}$ independent Bernoulli variables, each with success probability at least q. Thus, we can apply Chernoff's Inequality to X_v and find that

$$\mathbb{P}[X_v - \mathbb{E}[X_v] < -a] < e^{-2a^2/\hat{s}}$$

for any positive a. Since $\mathbb{E}[X_v] \geq q\hat{s}$, taking $a = (1 - \frac{1}{(1+\epsilon)})q\hat{s}$ we obtain

$$\mathbb{P}\left[X_v < \frac{1}{1+\epsilon}q\hat{s}\right] < \exp\left(\frac{-2((1-\frac{1}{1+\epsilon})q\hat{s})^2}{\hat{s}}\right).$$

Since $\hat{s} \geq s - r/m$, the inequality $s \geq \frac{r}{m} + \frac{\ln n}{2(1-1/(1+\epsilon))^2 q^2}$ implies that $\hat{s} \geq \frac{\ln n}{2(1-1/(1+\epsilon))^2 q^2}$. Thus, we may simplify:

$$\begin{aligned}\frac{2((1-\frac{1}{1+\epsilon})q\hat{s})^2}{\hat{s}} &= -2((1-\frac{1}{1+\epsilon})q)^2\hat{s} \\ &\geq \ln n.\end{aligned}$$

Therefore,

$$\mathbb{P}\left[X_v < \frac{1}{1+\epsilon}q\hat{s}\right] < e^{-\ln n} = 1/n.$$

Say that v is *unhappy* if $X_v < \frac{1}{1+\epsilon}q\hat{s}$. Taking the union bound over all v, we see that

$$\mathbb{P}[\text{some vertex is unhappy}] < 1.$$

It follows that with positive probability, every vertex v satisifies $X_v \geq \frac{1}{1+\epsilon}q\hat{s}$. The inequality $s \geq \frac{1+\epsilon}{q}\frac{r}{m}$ implies that $\frac{1}{1+\epsilon}q\hat{s} \geq \frac{\hat{r}}{m}$. Thus, there is some move for Phase 2 involving only the free spies (who have not yet moved) that guarantees $\hat{s}_{N[v]} \geq \frac{\hat{r}}{m}$ for all $v \in V(G)$. Since all meetings were covered in Phase 1, the new position is stable. $\square$

To complete the proof of Theorem 5.7.1, we must show that the spies can initially establish a stable position after the revolutionaries' initial placement.

The initial placement can be found randomly, using the same ideas as in Lemma 5.6.2. First we place one spy on each vertex hosting a meeting, and then we pick out some arbitrary vertex $v \in V(G)$ and place the remaining spies (as free spies) randomly among these vertices, placing each spy on a vertex chosen uniformly at random, with all decisions made independently. The resulting distribution of the spies is the same distribution produced by an equal number of initially-free spies moving randomly from v; thus, there is positive probability of obtaining a stable position, which we use as our initial placement.

We finish the section with the deferred proof of Lemma 5.7.3.

Proof of Lemma 5.7.3. We divide the proof into three steps. In Step 1, we prove that almost surely, every vertex v has degree close to its expectation. In Step 2, we prove that almost surely, $|N(v) \cap N(w)|$ is close to its expectation for all $v, w \in V(G)$. In Step 3, we show that G is $(p - \epsilon)$-common.

Step 1. We show that for any $\gamma > 0$, almost surely $d(v)$ is in the interval $((1-\gamma(n-1)p, (1+\gamma)(n-1)p)$ for all $v \in V(G)$. Call this interval I. Say that a vertex is *happy* if its degree is in I, and *unhappy* otherwise.

For any fixed n and any vertex v in $G(n,p)$, the degree $d(v)$ is a binomial random variable with $n-1$ trials and success probability p. Thus, Chernoff's Inequality applies, and we have

$$\begin{aligned}\mathbb{P}[v \text{ is unhappy}] = \mathbb{P}\left[d(v) - \mathbb{E}[d(v)] \geq \gamma(n-1)\right] &\leq 2e^{-2(\gamma(n-1)p)^2/(n-1)} \\ &= 2e^{-2\gamma^2 p^2 (n-1)}.\end{aligned}$$

By the union bound,

$$\mathbb{P}[\text{there is an unhappy vertex}] \leq 2ne^{-2\gamma^2 p^2 (n-1)}.$$

Since $\gamma^2 p^2$ is a positive constant, the right side goes to 0 as $n \to \infty$. Thus, almost surely all vertices are happy.

Step 2. We show that for any $\gamma > 0$, almost surely $|N(v) \cap N(w)|$ is in the interval $((p^2 - \gamma^2)(n-2), (p^2 + \gamma^2)(n-2))$ for all distinct $v, w \in V(G)$. Call this interval I. Say that a pair $\{v, w\}$ is *happy* if $|N(v) \cap N(w)|$ is in I, and *unhappy* otherwise.

For any fixed n and any distinct vertices $v, w \in V(W)$, let $X_{v,w} = |N(v) \cap N(w)|$. The variable $X_{v,w}$ is a binomial random variable with $n-2$ trials and success probability p^2. Thus, Chernoff's Inequality applies, and we have

$$\begin{aligned}\mathbb{P}[\{v, w\} \text{ is unhappy}] = \mathbb{P}\left[X_v - \mathbb{E}[X_v] \geq \gamma^2(n-2)\right] &\leq 2e^{-2(\gamma^2(n-2)p)^2/(n-2)} \\ &= 2e^{-2\gamma^4 p^2 (n-2)}.\end{aligned}$$

By the union bound,

$$\mathbb{P}[\text{there is an unhappy pair}] \leq 2\binom{n}{2}e^{-2\gamma^4 p^2(n-2)}.$$

Since $\gamma^4 p^2$ is a positive constant, the right side goes to 0 as $n \to \infty$. Thus, almost surely all pairs are happy.

Step 3. We use the previous two steps to show that G is $(p-\epsilon)$-common. Assume that G has no unhappy vertices and no unhappy pairs. For any distinct $v, w \in V(G)$, we have

$$\frac{|N(v) \cap N(w)|}{N(v)} \geq \frac{(p^2-\gamma^2)(n-2)}{(p+\gamma)(n-1)} = (p-\gamma)\frac{n-2}{n-1} = (p-\frac{\epsilon}{2})\frac{n-2}{n-1}.$$

When n is sufficiently large in terms of ϵ, we have $(p-\frac{\epsilon}{2})\frac{n-2}{n-1} \geq p-\epsilon$, so G is $(p-\epsilon)$-common. □

5.8 Spies on a Split Graph

Our canonical example of a graph that is good for revolutionaries, as given in Section 5.2, was a split graph. This suggests that we should study split graphs more broadly. As it turns out, when we restrict the degree of vertices in the split graph, the spies can use the structure of split graphs to their advantage.

The following theorem was stated in [4], with a promise that its proof would appear in this thesis:

Theorem 5.8.1. *Let G be a connected split graph with clique Q and independent set S in which each vertex of S has degree at most d. If m is a multiple of d, then $\sigma(G,m,r) \leq d\lceil r/m \rceil$.*

This statement of the theorem is weaker than it needs to be; the divisibility condition is unnecessary. We will actually prove the following theorem:

Theorem 5.8.2. *Let G be a connected split graph with clique Q and independent set S. If each vertex of S has degree at most d, where $d \geq 1$, then $\sigma(G,m,r) \leq \lceil dr/m \rceil$.*

Note that Theorem 5.8.2 is nearly sharp: when G is the split graph constructed in Theorem 5.2.2 for $m_0 = m$, the hypotheses of the theorem hold with $d = m$, so Theorem 5.8.2 gives the upper bound $\sigma(G,m,r) \leq r$, in contrast to the actual value $\sigma(G,m,r) = r-m+1$.

For the rest of this section, we will assume that G, d, and m are fixed satisfying the hypothesis of Theorem 5.8.2, and that $s = \lceil dr/m \rceil$. Order the vertices in Q arbitrarily. For each $v \in Q$, define the set $c(v)$ by

$$c(v) = \{v\} \cup \{w \in S : v = \min N(w)\}.$$

We think of each Q-vertex v as being "responsible for" all the vertices in $c(v)$. The sets $c(v)$ partition $V(G)$, since every S-vertex is adjacent to at least one Q-vertex.

Say that a position is *stable* if:

- all meetings are covered,
- all free spies are in Q, and
- $s_{c(v)} \geq \left\lfloor \frac{dr_{c(v)}}{m} \right\rfloor$ for all $v \in Q$.

Lemma 5.8.3. *If a position at the beginning of a round is stable, then after any move by the revolutionaries, the spies can respond to form a stable position at the end of the round.*

Proof. We use a Hall's Theorem argument, applied to an auxiliary bipartite graph whose partite sets represent spies and spy destinations. Form a set X as follows:

- Add to X one copy of each vertex hosting a meeting.
- For each $v \in Q$, add enough copies of v to X so that X contains at least $\left\lfloor \frac{dr'_{c(v)}}{m} \right\rfloor$ copies of vertices in $c(v)$.

Let Y be the set of spies. Construct an X,Y-bigraph H by making $x \in X$ and $y \in Y$ adjacent when the spy y can reach the vertex x in one move. We show that there is a matching that covers X. If we find such a matching then we are happy: sending each matched spy to the corresponding vertex covers all meetings and guarantees $s'_{c(v)} \geq \left\lfloor \frac{dr'_{c(v)}}{m} \right\rfloor$, and any unmatched spies in S can be sent to arbitrary vertices in Q. To apply Hall's Theorem, consider $X_0 \subseteq X$; we show that $|N_H(X_0)| \geq |X_0|$. Let $T \subseteq V(G)$ denote the set of vertices in $V(G)$ whose copies appear in X_0. We split into two cases: either T contains some Q-vertex, or $T \subseteq S$.

Case 1: *T contains some Q-vertex.* If T contains some $v \in Q$, then $N_G[v] \supseteq Q$, and so $N_H(X_0)$ contains every spy except for the ones that started the round covering meetings in $S - N_G[T]$. Let p denote the number of meetings in $S - N_G[T]$ at the beginning of the round; we have $|N_H(X_0)| = s - p$. We will show that $|X_0| \leq s - p$.

Let $U = \bigcup_{v \in T \cap Q} c(v)$, and let $q = |T - U|$. Note that $T - U \subseteq S$, so every meeting counted by q has only one copy in X_0. We claim that $r'_U \leq r - mp - mq$. To show this, first observe that at least mp revolutionaries started the round in meetings in $S - N_G[T]$; these revolutionaries cannot reach U, since S is independent and they are not adjacent to any vertex of $T \cap Q$. The mq revolutionaries that ended the round in meetings in $T - U$ also do not end the round in U. Since $T - U$ and $S - N_G[T]$ are disjoint subsets of the independent set S, a revolutionary starting the round in $S - N_G[T]$ cannot reach $T - U$ in a single round. Hence, there is no overlap between the mp revolutionaries that started the round in $S - N_G[T]$ and the mq revolutionaries that ended the round in $T - U$. This yields $r'_U \le r - mp - mq$. It follows that

$$\sum_{v \in T \cap Q} \left\lfloor \frac{dr'_{c(v)}}{m} \right\rfloor \le \frac{d}{m} \sum_{v \in T \cap Q} r'_{c(v)} = \frac{d}{m} r'_U \le \frac{d}{m}(r - mp - mq) \le \frac{dr}{m} - p - q \le s - p - q.$$

Now, to get the desired result, we observe that the sum

$$\sum_{v \in T \cap Q} \left\lfloor \frac{dr'_{c(v)}}{m} \right\rfloor$$

already counts the spies needed at every vertex of T *except* for the meetings in $T - U$. By hypothesis there are q such meetings, and each requires one spy, so we conclude that

$$|X_0| \le \left(\sum_{v \in T \cap Q} \left\lfloor \frac{dr'_{c(v)}}{m} \right\rfloor \right) + q \le s - p = |N_H(X_0)|,$$

which completes the case $T \cap Q \neq \emptyset$.

Case 2: $T \subseteq S$. In this case, every vertex appearing in X_0 appears only once – we can only have multiple copies of Q-vertices. We claim that

$$|X_0| \le \sum_{v \in N_G(T)} \left\lfloor \frac{dr_{c(v)}}{m} \right\rfloor \tag{5.1}$$

To establish this inequality, observe that for any $w \in T$, we have $N_G[w] \subseteq \bigcup_{v \in N(w)} c(v)$. Since $|N(w)| = d$, the Pigeonhole Principle says that there must be some $v \in N(w)$ such that the vertices in $c(v)$ sent at least m/d revolutionaries to the meeting at w (when $w \in c(v)$, this count includes any revolutionaries that started the round at w). For each $w \in T$, pick out such a vertex v, and say that v is *marked by* w.

Now a vertex that is marked i times must contribute at least i to the sum in Inequality (5.1): it contains i groups of at least m/d revolutionaries, each group destined for a different vertex in T. Since each $w \in T$ marks one vertex, it follows that the the sum in Inequality (5.1) must be at least $|T|$, which is equal to $|X_0|$. Since the position was stable at the beginning of the round, this yields

$$|X_0| \le \sum_{v \in N_G(T)} s_{c(v)} \le |N_H(X_0)|.$$

Hence Hall's Condition is satisfied. ☐

Theorem 5.8.4. *The spies win* $\mathrm{RS}(G, m, r, s)$.

Proof. It suffices to show that the spies can establish a stable position at the end of every round, since a stable position has all meetings covered. Since the above lemma allows the spies to re-establish a stable position whenever the starting position is stable, it suffices to show that the spies can form a stable position in response to the revolutionaries' initial placement.

To handle this, the spies can imagine a fictitious first revolutionary move where all revolutionaries start out on a single Q-vertex q. By putting all spies on q, the spies obtain a stable position:

- Every meeting is covered, since if $r \geq m$, then there is at least one spy, and that spy is on the same vertex as every revolutionary.
- All free spies are in Q, since all spies are in Q.
- $r_{c(v)} = 0$ when $v \neq q$, and

$$s_{c(q)} = s \geq \lceil dr/m \rceil \geq \lfloor dr/m \rfloor = \left\lfloor dr_{c(q)}/m \right\rfloor.$$

The imagined revolutionaries can then spend two turns moving to their real starting locations, and the spies can respond according to the lemma to maintain a stable position. □

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